Introduction to Mean Curvature Flow

LSGNT course, fall 2017

Version:
October 4, 2017

Felix Schulze,
University College London

Contents

1 Background

1.1 Geometry of Hypersurfaces

We give an introduction to the geometry of hypersurfaces in Euclidean space. For a more detailed background, we recommend [10, Chapter 6] and [25, §7].

We restrict ourselves to manifolds of codimension 1 in an Euclidean ambient space, i.e. we consider a n-dimensional smooth manifold M, without boundary, either closed or complete and non-compact and an immersion (or embedding)

$$F : M \to \mathbb{R}^{n+1}.$$

We call the image $F(M)$ a hypersurface. We will often identify points on M with their image under the immersion, if there is no risk of confusion.

Let $x = (x_1, \ldots, x_n)$ be a local coordinate system on M. The components of a vector v in the given coordinate system are denoted by v^i, the ones of a covector w are w_i. Mixed tensors have components with upper and lower indices depending on their type. We denote by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_e$$

the induced metric on M, where $\langle \cdot, \cdot \rangle_e$ is the Euclidean scalar product on $\mathbb{R}^{n+1}$. Note that the metric g induces anatural isomorphism between the tangent and the cotangent space. In coordinates, this is expressed in terms of raising/lowering indexes by means of the matrcies g_{ij} and g^{ij}, where g^{ij} is the inverse of g_{ij}. The scalar product on the tangent bundle naturally extends to any tensor bundle. For instance the scalar product of two $(1,2)$-tensors T^i_{jk} and S^i_{jk} is defined by

$$\langle T^i_{jk}, S^i_{jk} \rangle = T_i^{jk} S^i_{jk} = T^l_{pq} S^i_{jk} g_{li} g^{pj} g^{qk}\,.$$

The norm of a tensor T is then given by $|T| = \sqrt{\langle T, T \rangle}$. The volume element $d\mu$ (which is just the restriction of the n-dimensional Hausdorff measure to M), is given in local coordinates by

$$d\mu = \sqrt{\det g_{ij}}\, dx$$

Recall that on the ambient space $\mathbb{R}^{n+1}$ we have the standard covariant derivative $\bar{\nabla}$ given via directional derivatives of each coordinate, i.e. for two smooth vectorfields on X, Y on $\mathbb{R}^{n+1}$ we have

$$\bar{\nabla}_X Y\Big|_p = (D_{X(p)}Y^1(p), \cdots, D_{X(p)}Y^{n+1}(p))$$

where $Y(p) = (Y^1(p), \cdots, Y^{n+1}(p))$, and $D_{X(p)}$ is the directional derivative at p in direction $X(p)$. Recall that to define $D_{X(p)}Y^i(p)$ it is only necessary to locally know Y along an integral curve to X through p. Given two vectorfields V, W along $F(M)$ and tangent to M we thus define the connection

$$\nabla_V W := (\bar{\nabla}_V W)^T,$$

where ^{T} is the projection to the tangent space of M. One can check that this is the Levi-Civita connection corresponding to the induced metric g. In coordinates we obtain for the derivative of a vector v^i or a covector w^i the formulas

$$\nabla_k v^i = \frac{\partial v^i}{\partial x_k} + \Gamma^i_{jk}v^j, \qquad \nabla_k w_j = \frac{\partial w_j}{\partial x_k} - \Gamma^i_{jk}w_i,$$

where Γ^i_{jk} are the Christoffel symbols of the the connection ∇. This covariant derivative extends to tensors of all kind, in coordinates, we have e.g. for a (1,2)-tensor T^i_{jl}:

$$\nabla_k T^i_{jl} = \frac{\partial T^i_{jl}}{\partial x_k} + \Gamma^i_{mk}T^m_{jl} - \Gamma^m_{jk}T^i_{ml} - \Gamma^m_{kl}T^i_{jm}\ ,.$$

If f is a function, we set $\nabla_k f = \frac{\partial f}{\partial x_k}$, which concides with the differential $df\left(\frac{\partial}{\partial x_k}\right)$. Using the isomorphism induced by the metric g we can regard ∇f also as element of the tangent space, in this case it is called the *gradient* of f. The gradient of f can be identified with a vector in $\mathbb{R}^{n+1}$ via the differential dF; such a vector is called the *tangential gradient* of f and is denoted by $\nabla^M f$, given in coordinates by

$$\nabla^M f = \nabla^i f \frac{\partial F}{\partial x_i} = g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial F}{\partial x_i}\ .$$

The word "tangential" comes from the equivalent definition of $\nabla^M f$ in case f is a function defined on the ambient space $\mathbb{R}^{n+1}$. It can be checked that $\nabla^M f$ is the projection of the standard Euclidean gradient DF onto the tangent space of M, that is

$$\nabla^M f = Df - \langle Df, \nu\rangle_e\, \nu$$

where ν is a local choice of unit normal to M.

For two tangential vectorfields V, W, the shape operator is given by

$$S_V W = (\bar{\nabla}_V W)^{\perp}$$

where $^{\perp}$ is the projection to the normal space of M. Thus we have

$$\bar{\nabla}_V W = \nabla_V W + S_V W \ .$$

For local choice of unit normal vector field ν, the second fundamental form of M, a $(0,2)$-tensor, is given by

$$A(V, W) = -\langle S_V W, \nu \rangle_e = \langle W, \bar{\nabla}_V \nu \rangle_e \ ,$$

or in coordinates $A = (h_{ij})$ by

$$h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x_j \partial x_i}, \nu \right\rangle_e = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j} \nu \right\rangle_e \ .$$

The matrix of the Weingarten map $W(X) = \bar{\nabla}_X \nu : T_p M \to T_p M$ is given by $h^i_j = g^{il} h_{lj}$. The *principal curvatures* of M at a point are the eigenvalues of the symmetric matrix h^i_j, or equivalently the eigenvalues of h_{ij} with respect to g_{ij}. We denote the principal curvatures by $\lambda_1 \leq \cdots \leq \lambda_n$. The *mean curvature* is defined as the trace of the second fundamental form, i.e.

$$H = h^i_i = g^{ij} h_{ij} = \lambda_1 + \ldots + \lambda_n \ .$$

The square of the norm of the second fundamental form will be denoted by

$$|A|^2 = g^{mn} g^{st} h_{ms} h_{nt} = h^n_s h^s_n = \lambda_1^2 + \ldots + \lambda_n^2 .$$

It is easy to see that $|A|^2 \geq H^2/n$, with equality only if all the curvatures coincide; in fact we have the identity

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2 \ .$$

Clearly, A, W, H depend on the choice of orientation; if ν is reversed, their sign changes. But note that the *mean curvature vector*

$$\vec{H} = -H\nu$$

is independent of the orientation; in particular it is well defined globally even if

M is non-orientable.

We will call a hypersurface *convex* if the principal curvatures are non-negative everywhere. Observe that, with these definitions, if $F(M)$ is the boundary of a convex set, and the normal is outward pointing, then all principal curvatures are non-negative.

Recall the curvature tensor

$$R(X,Y,Z,W) = g(\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]}W, Z)$$

for vectorfields X,Y,Z,W on M.The Gauss equations relate the Riemann w.r.t. g to the curvature tensor of the ambient space in terms of the second fundamental form. Since the Euclidean ambient space is flat, we obtain

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}\,.$$

Thus the scalar curvature is given by

$$R = g^{ik}g^{jl}R_{ijkl} = H^2 - |A|^2 = 2\sum_{i<j}\lambda_i\lambda_j\ .$$

We also recall the *Codazzi equations*, which say that

$$\nabla_i h_{jk} = \nabla_j h_{ik}\,, \qquad i,j,k \in \{1,\ldots,n\},$$

i.e. taking into account the symmetry of h_{ij}, this implies that the tensor $\nabla A = \nabla_i h_{jk}$ is totally symmetric.

Let $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, i.e. an ambient vectorfield with compact support. Let $(\phi_t)_{-\varepsilon<t<\varepsilon}$ be the associated family of diffeomorphisms, i.e.

$$\frac{\partial \phi_t}{\partial t} = X(\phi_t)\,, \qquad \phi_0 = \mathrm{id}\ .$$

We then obtain a one-parameter family of variations of $F(M)$ via $\phi_t(F(M))$. We compute the variation of the measure as

$$\begin{aligned}\frac{\partial d\mu}{\partial t}\Big|_{t=0} &= \frac{\partial\sqrt{\det g_{ij}}}{\partial t}\Big|_{t=0} dx = \frac{1}{\sqrt{\det g_{ij}}}(\det g_{ij})g^{rs}\left\langle \frac{\partial X}{\partial x_r}, \frac{\partial F}{\partial x_s}\right\rangle_e dx \\ &= g^{rs}\left\langle \bar\nabla_{\frac{\partial F}{\partial x_r}} X, \frac{\partial F}{\partial x_s}\right\rangle_e d\mu\,,\end{aligned} \tag{1.1}$$

which leads us to define the tangential divergence

$$\operatorname{div}^M X = g^{ij} \left\langle \bar\nabla_{\frac{\partial F}{\partial x_i}} X, \frac{\partial F}{\partial x_j} \right\rangle_e = \sum_{i=1}^n \langle \bar\nabla_{e_i} X, e_i \rangle_e$$

where $e_1, \cdots, e_n$ is an ON-basis of T_pM. Recall the divergence theorem on a closed manifold

$$\int_M \operatorname{div}^M(X)\, d\mu = 0\,, \tag{1.2}$$

for $X \in \operatorname{Vec}_c(M)$. This follows directly from Stokes' theorem. For the normal part of a non-tangential vector field, one obtains

$$\begin{aligned}\operatorname{div}^M(X^\perp) &= \operatorname{div}^M(\langle X, \nu\rangle_e\, \nu) = \langle \nabla^M \langle X, \nu\rangle_e, \nu\rangle_e + \langle X, \nu\rangle_e \operatorname{div}^M \nu \\ &= \langle X, \nu\rangle_e g^{ij} \left\langle \bar\nabla_{\frac{\partial F}{\partial x_i}} \nu, \frac{\partial F}{\partial x_j} \right\rangle_e = \langle X, \nu\rangle_e g^{ij} h_{ij} = \langle X, \nu\rangle_e H = -\langle X, \vec{H}\rangle_e\end{aligned}$$

Together with (1.2) this yields the general divergence theorem

$$\int_M \operatorname{div}^M(X)\, d\mu = \int_M \operatorname{div}^M(X^T) + \operatorname{div}^M(X^\perp)\, d\mu = -\int_M \langle X, \vec{H}\rangle_e\, d\mu\,, \tag{1.3}$$

for $X \in \operatorname{Vec}_c(\mathbb{R}^{n+1})$. Together with (1.1) this yields the first variation formula

$$\frac{\partial}{\partial t}\Big|_{t=0} \int_{\phi_t(M)} 1\, d\mu_t = \int_M \operatorname{div}^M(X)\, d\mu = -\int_M \langle X, \vec{H}\rangle_e\, d\mu\,. \tag{1.4}$$

We recall the *Laplace-Beltrami operator* on functions $f : M \to \mathbb{R}$ given by

$$\Delta^M f = \operatorname{div}^M(\nabla^M f)\ .$$

We write simply Δ instead of Δ^M. One can easily check that

$$\Delta^M f = g^{ij}\nabla_i\nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial f}{\partial x_k}\right) = \frac{1}{\sqrt{\det g_{ij}}}\frac{\partial}{\partial x_i}\left(\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial f}{\partial x_j}\right)\ .$$

The divergence theorem then gives the usual integration by parts formula

$$\int_M f\Delta h\, d\mu = -\int_M \langle \nabla f, \nabla h\rangle\, d\mu = \int_M h\Delta f\, d\mu\ .$$

If f is a function on the ambient space we have by the above calculations

$$\begin{aligned}\Delta^M f &= \mathrm{div}^M(\nabla^M f) = \mathrm{div}^M(Df) - \mathrm{div}^M(Df^\perp)\\ &= \Delta^{\mathbb{R}^{n+1}} f - D^2 f(\nu,\nu) + \langle Df, \vec{H}\rangle_e\ .\end{aligned} \tag{1.5}$$

Thus Δ^M not only neglects the contribution of the second derivatives normal to M, but also takes into account the curvature of M.

Let $X = (x_1, \ldots, x_{n+1})$ be the coordinates of $\mathbb{R}^{n+1}$. Equation (1.5) yields

$$\Delta^M x_i = \langle \vec{H}, e_i\rangle_e$$

where e_i is the i-th basis vector of $\mathbb{R}^{n+1}$. We can thus write

$$\Delta^M X = \vec{H}\ .$$

Note that in coordinates the vectorfield X is just given by F, and we can write

$$\Delta^M F = \vec{H}\ .$$

We also note the identity

$$\Delta^M |X|_e^2 = 2n + 2\langle X, \vec{H}\rangle_e\,. \tag{1.6}$$

The second fundamental form corresponds in a certain sense to second derivatives of an immersion, and its symmetry reflects that second partial derivatives of a function commute. Similarly the Codazzi equations can be seen as a geometric manifestation that third partial derivatives commute. Thus we can also expect that there is a symmetry of the second covariant derivatives of the second fundamental form. This identity is known as *Simon's identity*:

$$\nabla_k\nabla_l h_{ij} = \nabla_i\nabla_j h_{kl} + h_{kl}h_i^{\ m}h_{mj} - h_{km}h_{il}h^m_{\ j} + h_{kj}h_i^{\ m}h_{ml} - h_k^{\ m}h_{ij}h_{ml} \tag{1.7}$$

For a proof see [23]. We note the following two consequences

$$\Delta h_{ij} = \nabla_i\nabla_j H + H h_i^{\ m}h_{mj} - h_{ij}|A|^2 \tag{1.8}$$

and

$$\frac{1}{2}\Delta|A|^2 = h^{ij}\nabla_i\nabla_j H + |\nabla A|^2 + H\mathrm{tr}(A^3) - |A|^4\ . \tag{1.9}$$

We give the explicit expressions of the main geometric quantities in the case when $F(M)$ is the graph of a function $x_{n+1} = u(x_1, \ldots, x_n)$. We choose the orientation where ν points downwards. By straightforward computations one gets

$$\nu = \frac{(D_1u, \ldots, D_nu, -1)}{\sqrt{1+|Du|^2}} \,, \tag{1.10}$$

$$g_{ij} = \delta_{ij} + D_iuD_ju, \qquad g^{ij} = \delta_{ij} - \frac{D_iuD_ju}{1+|Du|^2} \,, \tag{1.11}$$

$$h_{ij} = \frac{D^2_{ij}u}{\sqrt{1+|Du|^2}}, \qquad H = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \,, \tag{1.12}$$

where div is the standard divergence on $\mathbb{R}^n$.

1.2 Maximum principles

We will need the following maximum principles. The first one is the standard maximum principle for scalar functions:

Theorem 1.2.1 (Strong maximum principle for parabolic equations)**.**
Let M be closed and $f : M \times [0,T) \to \mathbb{R}$ satisfy

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + c\, f$$

for some smooth funtions b^i, c, where $c \geq 0$. If $f(\cdot, 0) \geq 0$ then

$$\min_M f(\cdot, t) \geq \min_M f(\cdot, 0) \,.$$

Furthermore, if $f(p, t_0) = \min_M f(\cdot, 0)$ for some $p \in M$, $t > 0$, then $f \equiv \min_M f(\cdot, 0)$ for $0 \leq t \leq t_0$.

For a proof see for example [14, Chapter 6.4 and Chapter 7.1.4] . The maximum principle can be extended to symmetric 2-tensors:

Theorem 1.2.2 (Strong parabolic maximum principle for symmetric 2-tensors (Hamilton))**.** *Let M be closed and $m^i_{\;j}$ be a symmetric bilinear form, which solves*

$$\frac{\partial m^i_{\;j}}{\partial t} \geq \Delta m^i_{\;j} + \phi^i_{\;j}(m^i_{\;j}) \,,$$

where $\phi^i_{\;j}$ is a symmetric bilinear form, depending on $m^i_{\;j}$, with the property

$\phi^i_j(m^i_j) \geq 0$ if $m^i_j \geq 0$. If $m^i_j \geq 0$ for $t = 0$ then $m^i_j \geq 0$ for all $t \geq 0$. Furthermore, for $t > 0$, the rank of the null-space of m^i_j is constant, and the null-space is invariant under parallel transport and invariant in time.

For a proof see [17, Lemma 8.2]. It is helpful to think about m^i_j being in diagonal form and applying the parabolic scalar maximum principle to the smallest eigenvalue (there is actually a way to prove the maximum principle using this idea - one needs to find a way how to approximate the minimum of n functions in a smooth way preserving convexity).

We also note the strong elliptic maximum principle:

Theorem 1.2.3 (Strong elliptic maximum principle)**.** *Let M be closed and $f : M \to \mathbb{R}$ satisfy*

$$-\Delta f + b^i \nabla_i f + c\, f \leq 0$$

for some smooth funtions b^i, c, where $c \geq 0$. If $f \leq 0$, but $f \neq 0$, then $f < 0$.

For a proof see [14, §6.4, Theorem 4].

2 Basic properties

Let M^n be closed (or non-compact and complete), and $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth family of immersions. Let $M_t := F(M,t)$. We call this family a Mean Curvature flow, MCF for short, starting at an initial immersion F_0, if

$$\begin{aligned} \frac{\partial F}{\partial t} &= -H \cdot \nu = \vec{H} \quad (= \Delta_{M_t} F) \\ F(\cdot, 0) &= F_0 \ . \end{aligned} \tag{2.1}$$

Remark 2.0.1: i) In general, it suffices to ask that

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} \ .$$

One solves the ODE on M given by

$$\frac{\partial \phi}{\partial t} = -dF^{-1}\left(\left(\frac{\partial F}{\partial t}\right)^{T}\right)(\phi)$$

with $\phi(0) = \mathrm{id}$. Then $\tilde{F} := F \circ \phi$ solves usual MCF.

ii) The evolution equation for a surface, which is locally given as the graph of a function u, is thus

$$\left(\frac{\partial u}{\partial t}\, e_{n+1}\right)^{\perp} = \vec{H}$$

or equivalently

$$\frac{\partial u}{\partial t}\, \langle e_{n+1}, \nu \rangle = -H \ ,$$

which yields

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\ \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_i D_j u \, . \tag{2.2}$$

This is a quasilinear parabolic equation.

iii) By formula (1.3) we have for an evolution with normal speed $-f\nu$ that

$$\frac{d}{dt}|M_t| = \frac{d}{dt}\int_M 1\, d\mu_t = -\int_M fH\, d\mu\,,$$

and thus by the Hölder's inequality, mean curvature flow decreases area the fastest, when comparing with speeds with the same L^2-norm.

Examples: There are not many explicit examples of MCF solutions.

i) The most basic one is the evolution of a sphere with initial radius $R > 0$. Assuming that the solutions remains rotationally symmetric (which follows from uniqueness, see later), we obtain the following ODE for the radius $r(t)$:

$$\frac{\partial r}{\partial t} = -\frac{n}{r}.$$

with initial condition $r(0) = R$. Integrating yields $r(t) = \sqrt{R^2 - 2nt}$. Note that the maximal existence time $T = R^2/(2n)$ is finite and the curvature blows up for $t \to T$. Furthermore, the shrinking sphere is an example of a solution which only moves by scaling, a so-called *self-similar shrinker*.

By the previous example the evolution of a cylinder

$$\mathbb{S}^k_R \times \mathbb{R}^{n-k}$$

remains cylindrical with radius given by $r(t) = \sqrt{R^2 - 2kt}$. Note that again this solution is *self-similarly shrinking*.

Another class of examples are translating solutions. Assuming that they translate with speed one in direction τ, they satisfy the elliptic equation

$$H = -\langle \tau, \nu \rangle.$$

Assuming that the solution is graphical, i.e. $x_{n+1} = u(x_1, \cdots, x_n)$, and moving in e_{n+1} direction we obtain from (2.2) that it satisfies the equation

$$\left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_i D_j u = 1\,.$$

In one dimension the equation becomes

$$y_{xx} = 1 + y_x^2$$

which can be integrated explicitly, yielding $y(x) = -\ln\cos x$ for $|x| < \pi/2$, up to translation and adding constants. This solution is usually called the *grim reaper*. In higher dimensions it can be shown that there is a unique, convex, rotationally symmetric solution - but which is defined on the whole space. For properties of this solution see [8]. For $n = 2$ these are the unique convex translating entire graphs, but for $n \geq 3$ there exist entire convex translating graphs which are not rotationally symmetric, see [26].

The upwards translating grim reaper given by $e^{-y(t)} = e^{-t}\cos x(t)$ and the downwards translating grim reaper given by $e^{y(t)} = e^{-t}\cos x(t)$ can be combined to give another pair of solutions given implicitly as the solution set of

$$\cosh y(t) = e^t \cos x(t)\ , \tag{2.3}$$

and

$$\sinh y(t) = e^t \cos x(t)\ . \tag{2.4}$$

The *paperclip*, given as solution of (2.3) restricted to $|x| < \pi/2$ desribes a compact *ancient* solution that for $t \to 0$ becomes extinct in a round point and for $t \to -\infty$ looks like two copies of the grim reaper glued together smoothly. The *hairclip* (2.4) is an *eternal* solution, which for $t \to -\infty$ looks like an infinite row of grim reapers, alternating between translating up and translating down, and for $t \to +\infty$ converges to a horizontal line.

We have the following short-time existence result.

Theorem 2.0.2 (Short-time existence)**.** *Let* $F_0 : M^n \to \mathbb{R}^{n+1}$ *be a smooth im-*

mersion of a closed n-dimensional manifold M. Then there exists a unique smooth solution on a maximal time interval $[0,T)$ for $T \in (0,\infty]$.

The difficulty to prove this result comes from the geometric nature of the flow, which makes any solution invariant under diffeomorphisms of M and thus the evolution equation is only weakly parabolic. There different ways to prove this result. One can either follow the approach of Hamilton [] for the Ricci flow and use the Nash-Moser Implicit function theorem. Alternatively one can use the so-called De Turck to break the diffeomeorphism invariance. The most natural way is to write the evolving surfaces $M_t = F(M,t)$ for a short time as an exponential normal graph over $M_0 = F_0(M)$. One can then check that the height function u satisfies a quasilinear parabolic equation similar to (2.2) for which standard results for those type of equations can be applied. For details see [23].

The strong maximum principle implies the following.

Theorem 2.0.3 (Avoidance principle). *Assume two solutions to mean curvature flow $(M_t^1)_{t\in[0,T)}$ and $(M_t^2)_{t\in[0,T)}$ are initially disjoint (and at least one of them is compact), i.e. $M_0^1 \cap M_0^2 = \emptyset$. Then $M_t^1 \cap M_t^2 = \emptyset \;\; \forall\, t \in (0,T)$.*

Proof. Assume that this is not the case. Then there exists a first time $t_0 \in (0,T)$ where $M_{t_0}^1$ and $M_{t_0}^2$ touch at the point $x_0 \in \mathbb{R}^{n+1}$. Note that this implies that $T_{x_0}M_{t_1}^1 = T_{x_0}M_{t_1}^2 := T$ and there is an $\varepsilon > 0$ such that we can write $(M_t^1)_{t_0-\varepsilon\le t\le t_0}$ and $(M_t^2)_{t_0-\varepsilon\le t\le t_0}$ locally as graphs over the affine space $x_0 + T$. The two graph functions u_1, u_2 satisfy (2.2) which we write as

$$\frac{\partial u}{\partial t} = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1+|Du|^2}\right) D_{ij}u =: a^{ij}(Du) D_{ij}u.$$

We can assume w.l.o.g that $u_2 \le u_1$ and $u_1 = u_2$ at (x_0,t_0). But note that

$v = u_1 - u_2$ satisfies a linear parabolic equation:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= a^{ij}(Du_1)D_iD_ju_1 - a^{ij}(Du_2)D_iD_ju_2 \\
&= \int\limits_0^1 \frac{d}{ds}\big(a^{ij}(D(su_1+(1-s)u_2)D_{ij}(su_1+(1-s)u_2)\big)\,ds \\
&= \Big(\int\limits_0^1 a^{ij}(D(su_1+(1-s)u_2))\,ds\Big)D_{ij}v \\
&\qquad + \Big(\int\limits_0^1 \frac{\partial a^{ij}}{\partial p_k}(D(su_1+(1-s)u_2))D_{ij}(su_1+(1-s)u_2)\,ds\Big)D^kv \\
&=: \tilde{a}^{ij}D_{ij}v + \tilde{b}^kD_kv\,,
\end{aligned}$$

where p is the Du variable of $a^{ij}(p)$. Note that $\tilde{a}^{ij}$ is symmetric and strictly positve. Since $v \geq 0$ and $v = 0$ at (x_0, t_0) the strong maximum principle implies that $v \equiv 0$ which yields a contradiction. □

With more or less the same argument one can show the following.

Corollary 2.0.4 (Preservation of embeddedness)**.** *If M_0 is closed and embedded, then M_t is embedded for all t.*

Remark 2.0.5: (i) Enclosing a compact initial hypersurface M_0 by a large sphere, and using that the maximal existence time of the evolution of the sphere is finite, we obtain that the maximal existence time T is finite.

(ii) Note the we can translate a solution to MCF in the ambient space and get a new solution to MCF. Thus the avoidance principle implies that the distance between two disjoint solutions is non-decreasing in time.

(iii) In case M_0 is embedded, we will always choose ν to be the *outward* unit normal.

2.0.1 Outline of the course

First, we will compute the evolution equations of the main geometric quantities and show for example that convexity and non-negative mean curvature are preserved. Then we will show that the flow exists smoothly as long as the second fundamental form stays bounded.

A main tool in the analysis of singularities is Huisken's monotonicity formula. We will derive it, and show that it implies that any tangent flow (if it exists) is a self-similarly shrinking solution. Following an argument of White [27], we will use the monotonicity formula to show that a control on the Gaussian density ratios implies a control on the curvature. We will conclude with the classification of mean convex self-similarly shrinking solutions and self-similarly shrinking curves in the plane.

For mean curvature flow of curves in the plane, the so-called *curve shortening flow*, the following theorem holds:

Theorem 2.0.6 (Gage/Hamilton [15], Grayson [16])**.** *Under curve shortening flow, simple, closed curves become convex in finite time and shrink to a 'round' point.*

We will not follow the original proof, but use Huisken's monotonicity formula and a quantitative control of embeddedness, which will rule out certain singularities.

In higher dimensions one cannot expect that such a behaviour is true, since one can rather easily construct counterexamples. But the following fundamental result of Huisken holds:

Theorem 2.0.7 (Huisken [20])**.** *Any closed, convex hypersurface becomes immediately strictly convex under MCF and converges in finite time to a 'round' point.*

We will give a proof of this result, making again strong use of the monotonicity formula.

The next part will focus on two-convex mean curvature flow, that is when $\lambda_1+\lambda_2 \geq 0$ everywhere on M_0, which we will see is preserved under the evolution. We will show that this implies that the only possible singularities are asymptotic either to a shrinking sphere or a shrinking cylinder with only *one* straight direction. We will then present the result of Huisken-Sinestrari that this structure allows one to define a mean curvature flow with surgery:

Theorem 2.0.8 (Huisken-Sinestrari [24]). *Let $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed n-dimensional hypersurface with $n \geq 3$. Assume M_0 is two-convex. Then there exists a mean curvature flow with surgeries starting from M_0 which terminates after a finite number of steps.*

The result has topological consequences, which we will also discuss. It is important to note that this is the extrinsic analogue of the results of Hamilton/Perelman on 3-dimensional Ricci flow with surgeries / through singularities.

Furthermore, we will also discuss the recent result Brendle [7] that the only embedded $M^2 \subset \mathbb{R}^3$ which is self-shrinking and diffeomorphic to $\mathbb{S}^2$ is round. If time permits, we are planning to discuss applications of mean curvature flow to prove optimal isoperimetric inequalities.

Here is list of further introductory texts on mean curvature flow (and which I have partially used and copied from in preparation of these notes):

- B. White, *Topics in mean curvature flow*, lecture notes by O. Chodosh. Available at `https://web.math.princeton.edu/~ochodosh/notes.html`

- K. Ecker, *Regularity theory for Mean Curvature Flow*, Birkhäuser

- M. Ritoré and C. Sinestrari, *Mean Curvature Flow and isoperimetric inequalities*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser

- C. Mantegazza, *Lecture Notes in Mean Curvature Flow*, Progress in Mathematics, Volume 290, Birkhäuser

- R. Haslhofer, *Lectures on curve shortening flow.* Available at `http://www.math.toronto.edu/roberth/pde2/curve_shortening_flow.pdf`

- R. Haslhofer, *Lectures on mean curvature flow.* Available at `https://`

`arxiv.org/abs/1406.7765`.

2.1 The maximal time of existence

We first compute the basic evolution equations.

Lemma 2.1.1. *The following evolution equations hold.*

(i) $\frac{\partial}{\partial t}\nu = \nabla H$ (ii) $\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij}$

(iii) $\frac{\partial}{\partial t}g^{ij} = 2Hh^{ij}$ (iv) $\frac{\partial}{\partial t}d\mu = -H^2\, d\mu$

(v) $\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{im}h^m_{\ j} + |A|^2 h_{ij}$ (vi) $\frac{\partial}{\partial t}h^i_{\ j} = \Delta h^i_{\ j} + |A|^2 h^i_{\ j}$

(vii) $\frac{\partial}{\partial t}H = \Delta H + |A|^2 H$ (viii) $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4$

Proof. (i) We first note that $\langle \nu, \nu\rangle \equiv 1$ so we obtain

$$\Big\langle \frac{\partial \nu}{\partial t}, \nu \Big\rangle = 0\,.$$

Since $\langle \nu, \frac{\partial F}{\partial x_i}\rangle \equiv 1$ we can compute

$$\Big\langle \frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x_i}\Big\rangle = -\Big\langle \nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_i}\Big\rangle = \Big\langle \nu, \frac{\partial}{\partial x_i}(H\nu)\Big\rangle = \frac{\partial H}{\partial x_i}\ ,$$

where we used that $\langle \frac{\partial}{\partial x_i}\nu, \nu\rangle = 0$. This yields the statement.

(ii) We have

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\Big\rangle = -\Big\langle \frac{\partial}{\partial x_i}(H\nu), \frac{\partial F}{\partial x_j}\Big\rangle - \Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(H\nu)\Big\rangle \\ &= -Hh_{ij}\end{aligned}$$

(iii) This follows from differentiating the identity

$$g^{il}g_{lj} = \delta^i_{\ j}\,.$$

(iv) This follows since by (1.1) and following calculation we have

$$\frac{\partial}{\partial t} d\mu = \operatorname{div}^M(\vec{H})\, d\mu = -\langle \vec{H}, \vec{H} \rangle\, d\mu = -H^2\, d\mu\,.$$

(v) We choose normal coordinates at (p,t). Note that this implies that all Christoffel symbols at that point vanish and the partial derivatives coincide with the covariant derivatives.

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij} &= \frac{\partial}{\partial t}\Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial x_j}\Big\rangle = -\Big\langle \frac{\partial}{\partial x_i}(H\nu), \frac{\partial \nu}{\partial x_j}\Big\rangle + \Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(\nabla H)\Big\rangle \\ &= -H h_{im} h^m_{\ j} + \nabla_j \nabla_i H\end{aligned}$$

Combining this with Simon's identity (1.8) yields

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{im} h^m_{\ j} + h_{ij}|A|^2$$

(vi) Follows from (v) combined with (iii).

(vii) Follows from (vi) by taking a trace.

(viii) Follows from (vi) by writing $|A|^2 = h^i_{\ j} h_i^{\ j}$ and noting that in normal coordinates at a point (p,t)

$$\Delta |A|^2 = \sum_l \nabla_l \nabla_l h^i_{\ j} h_i^{\ j} = h^i_{\ j} \Delta h_i^{\ j} + h_i^{\ j} \Delta h^i_{\ j} + 2|\nabla A|^2\ .$$

□

By the strong maximum principle we obtain the following two theorems.

Theorem 2.1.2. *Assume $M_0 = F_0(M)$ closed and mean convex, i.e. $H \geq 0$. Then $H > 0$ for all $t > 0$.*

Proof. That $H \geq 0$ for $t \geq 0$ follows from the evolution equation of H and the parabolic maximum principle, Theorem 1.2.1. Assume now that $H(p_0, t_0) = 0$ for some $t_0 > 0$. The strong maximum principle then implies that $H \equiv 0$ for all (p,t) and $0 \leq t \leq t_0$. But this is impossible since any closed hypersurface in $\mathbb{R}^{n+1}$ has points where $\lambda_1 > 0$. □

Theorem 2.1.3. *Assume* $M_0 = F_0(M)$ *closed and convex, i.e.* ${h^i}_j \geq 0$. *Then* ${h^i}_j > 0$ *for all* $t > 0$.

Proof. That ${h^i}_j \geq 0$ for $t \geq 0$ follows from the evolution equation of ${h^i}_j$ and the parabolic maximum principle for 2-tensors, Theorem 1.2.2. Assume now that ${h^i}_j(p_0, t_0)$ has a zero eigenvalue for some $t_0 > 0$. The strong maximum principle then implies that the rank of the null-space is greater or equal to one for all (p, t) and $0 \leq t \leq t_0$. But this again is impossible since there exist points where $\lambda_1 > 0$. □