Introduction
to Mean Curvature Flow

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Felix Schulze,
University College London

Contents

1 Background

1.1 Geometry of Hypersurfaces

We give an introduction to the geometry of hypersurfaces in Euclidean space. For a more detailed background, we recommend [11, Chapter 6] and [35, §7].

We restrict ourselves to manifolds of codimension 1 in an Euclidean ambient space, i.e. we consider a n-dimensional smooth manifold M, without boundary, either closed or complete and non-compact and an immersion (or embedding)

$$F : M \to \mathbb{R}^{n+1}.$$

We call the image $F(M)$ a hypersurface. We will often identify points on M with their image under the immersion, if there is no risk of confusion.

Let $x = (x_1, \dots, x_n)$ be a local coordinate system on M. The components of a vector v in the given coordinate system are denoted by v^i, the ones of a covector w are w_i. Mixed tensors have components with upper and lower indices depending on their type. We denote by

$$g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle_e$$

the induced metric on M, where $\langle \cdot, \cdot \rangle_e$ is the Euclidean scalar product on $\mathbb{R}^{n+1}$. Note that the metric g induces anatural isomorphism between the tangent and the cotangent space. In coordinates, this is expressed in terms of raising/lowering indexes by means of the matrcies g_{ij} and g^{ij}, where g^{ij} is the inverse of g_{ij}. The scalar product on the tangent bundle naturally extends to any tensor bundle. For instance the scalar product of two $(1,2)$-tensors T^i_{jk} and S^i_{jk} is defined by

$$\langle T^i_{jk}, S^i_{jk} \rangle = T_i^{jk} S^i_{jk} = T^l_{pq} S^i_{jk} g_{li} g^{pj} g^{qk} \,.$$

The norm of a tensor T is then given by $|T| = \sqrt{\langle T, T \rangle}$. The volume element $d\mu$ (which is just the restriction of the n-dimensional Hausdorff measure to M), is given in local coordinates by

$$d\mu = \sqrt{\det g_{ij}}\, dx$$

Recall that on the ambient space $\mathbb{R}^{n+1}$ we have the standard covariant derivative $\bar{\nabla}$ given via directional derivatives of each coordinate, i.e. for two smooth vectorfields on X, Y on $\mathbb{R}^{n+1}$ we have

$$\bar{\nabla}_X Y\Big|_p = (D_{X(p)}Y^1(p), \cdots, D_{X(p)}Y^{n+1}(p))$$

where $Y(p) = (Y^1(p), \cdots, Y^{n+1}(p))$, and $D_{X(p)}$ is the directional derivative at p in direction $X(p)$. Recall that to define $D_{X(p)}Y^i(p)$ it is only necessary to locally know Y along an integral curve to X through p. Given two vectorfields V, W along $F(M)$ and tangent to M we thus define the connection

$$\nabla_V W := (\bar{\nabla}_V W)^T\,,$$

where ^{T} is the projection to the tangent space of M. One can check that this is the Levi-Civita connection corresponding to the induced metric g. In coordinates we obtain for the derivative of a vector v^i or a covector w^i the formulas

$$\nabla_k v^i = \frac{\partial v^i}{\partial x_k} + \Gamma^i_{jk} v^j, \qquad \nabla_k w_j = \frac{\partial w_j}{\partial x_k} - \Gamma^i_{jk} w_i,$$

where Γ^i_{jk} are the Christoffel symbols of the the connection ∇. This covariant derivative extends to tensors of all kind, in coordinates, we have e.g. for a (1,2)-tensor T^i_{jl}:

$$\nabla_k T^i_{jl} = \frac{\partial T^i_{jl}}{\partial x_k} + \Gamma^i_{mk} T^m_{jl} - \Gamma^m_{jk} T^i_{ml} - \Gamma^m_{kl} T^i_{jm}\ ,.$$

If f is a function, we set $\nabla_k f = \frac{\partial f}{\partial x_k}$, which concides with the differential $df\left(\frac{\partial}{\partial x_k}\right)$. Using the isomorphism induced by the metric g we can regard ∇f also as element of the tangent space, in this case it is called the *gradient* of f. The gradient of f can be identified with a vector in $\mathbb{R}^{n+1}$ via the differential dF; such a vector is called the *tangential gradient* of f and is denoted by $\nabla^M f$, given in coordinates by

$$\nabla^M f = \nabla^i f \frac{\partial F}{\partial x_i} = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial F}{\partial x_i}\ .$$

The word "tangential" comes from the equivalent definition of $\nabla^M f$ in case f is a function defined on the ambient space $\mathbb{R}^{n+1}$. It can be checked that $\nabla^M f$ is the projection of the standard Euclidean gradient DF onto the tangent space of M, that is

$$\nabla^M f = Df - \langle Df, \nu \rangle_e\, \nu$$

where ν is a local choice of unit normal to M.

For two tangential vectorfields V, W, the shape operator is given by

$$S_V W = (\bar{\nabla}_V W)^{\perp}$$

where $^{\perp}$ is the projection to the normal space of M. Thus we have

$$\bar{\nabla}_V W = \nabla_V W + S_V W \ .$$

For local choice of unit normal vector field ν, the second fundamental form of M, a $(0,2)$-tensor, is given by

$$A(V,W) = -\langle S_V W, \nu\rangle_e = \langle W, \bar{\nabla}_V \nu\rangle_e \ ,$$

or in coordinates $A = (h_{ij})$ by

$$h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x_j \partial x_i}, \nu \right\rangle_e = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}\nu \right\rangle_e \ .$$

The matrix of the Weingarten map $W(X) = \bar{\nabla}_X \nu : T_pM \to T_pM$ is given by $h^i_j = g^{il}h_{lj}$. The *principal curvatures* of M at a point are the eigenvalues of the symmetric matrix h^i_j, or equivalently the eigenvalues of h_{ij} with respect to g_{ij}. We denote the principal curvatures by $\lambda_1 \leq \cdots \leq \lambda_n$. The *mean curvature* is defined as the trace of the second fundamental form, i.e.

$$H = h^i_i = g^{ij}h_{ij} = \lambda_1 + \ldots + \lambda_n \ .$$

The square of the norm of the second fundamental form will be denoted by

$$|A|^2 = g^{mn}g^{st}h_{ms}h_{nt} = h^n_s h^s_n = \lambda_1^2 + \ldots + \lambda_n^2.$$

It is easy to see that $|A|^2 \geq H^2/n$, with equality only if all the curvatures coincide; in fact we have the identity

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}(\lambda_i - \lambda_j)^2 \ . \tag{1.1}$$

Clearly, A, W, H depend on the choice of orientation; if ν is reversed, their sign changes. But note that the *mean curvature vector*

$$\vec{H} = -H\nu$$

is independent of the orientation; in particular it is well defined globally even if

M is non-orientable.

We will call a hypersurface *convex* if the principal curvatures are non-negative everywhere. Observe that, with these definitions, if $F(M)$ is the boundary of a convex set, and the normal is outward pointing, then all principal curvatures are non-negative.

Recall the curvature tensor

$$R(X,Y,Z,W) = g(\nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W, Z)$$

for vectorfields X, Y, Z, W on M.The Gauss equations relate the Riemann w.r.t. g to the curvature tensor of the ambient space in terms of the second fundamental form. Since the Euclidean ambient space is flat, we obtain

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}\,.$$

Thus the scalar curvature is given by

$$R = g^{ik}g^{jl}R_{ijkl} = H^2 - |A|^2 = 2\sum_{i<j}\lambda_i\lambda_j\ .$$

We also recall the *Codazzi equations*, which say that

$$\nabla_i h_{jk} = \nabla_j h_{ik}\,, \qquad i,j,k \in \{1,\ldots,n\},$$

i.e. taking into account the symmetry of h_{ij}, this implies that the tensor $\nabla A = \nabla_i h_{jk}$ is totally symmetric.

Let $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, i.e. an ambient vectorfield with compact support. Let $(\phi_t)_{-\varepsilon<t<\varepsilon}$ be the associated family of diffeomorphisms, i.e.

$$\frac{\partial\phi_t}{\partial t} = X(\phi_t)\,, \qquad \phi_0 = \mathrm{id}\ .$$

We then obtain a one-parameter family of variations of $F(M)$ via $\phi_t(F(M)$. We compute the variation of the measure as

$$\begin{aligned}\frac{\partial d\mu}{\partial t}\Big|_{t=0} &= \frac{\partial\sqrt{\det g_{ij}}}{\partial t}\Big|_{t=0} dx = \frac{1}{\sqrt{\det g_{ij}}}(\det g_{ij})g^{rs}\left\langle\frac{\partial X}{\partial x_r},\frac{\partial F}{\partial x_s}\right\rangle_e dx \\ &= g^{rs}\left\langle\bar\nabla_{\frac{\partial F}{\partial x_r}}X,\frac{\partial F}{\partial x_s}\right\rangle_e d\mu\,,\end{aligned} \tag{1.2}$$

which leads us to define the tangential divergence

$$\mathrm{div}^M X = g^{ij}\left\langle \bar{\nabla}_{\frac{\partial F}{\partial x_i}} X, \frac{\partial F}{\partial x_j}\right\rangle_e = \sum_{i=1}^n \langle \bar{\nabla}_{e_i} X, e_i\rangle_e$$

where $e_1, \cdots, e_n$ is an ON-basis of T_pM. Recall the divergence theorem on a closed manifold

$$\int_M \mathrm{div}^M(X)\, d\mu = 0\,, \tag{1.3}$$

for $X \in \mathrm{Vec}_c(M)$. This follows directly from Stokes' theorem. For the normal part of a non-tangential vector field, one obtains

$$\begin{aligned}\mathrm{div}^M(X^\perp) &= \mathrm{div}^M(\langle X,\nu\rangle_e\,\nu) = \langle \nabla^M\langle X,\nu\rangle_e, \nu\rangle_e + \langle X,\nu\rangle_e \mathrm{div}^M \nu\\ &= \langle X,\nu\rangle_e g^{ij}\left\langle \bar{\nabla}_{\frac{\partial F}{\partial x_i}}\nu, \frac{\partial F}{\partial x_j}\right\rangle_e = \langle X,\nu\rangle_e g^{ij}h_{ij} = \langle X,\nu\rangle_e H = -\langle X, \vec{H}\rangle_e\end{aligned}$$

Together with (1.3) this yields the general divergence theorem

$$\int_M \mathrm{div}^M(X)\, d\mu = \int_M \mathrm{div}^M(X^T) + \mathrm{div}^M(X^\perp)\, d\mu = -\int_M \langle X, \vec{H}\rangle_e\, d\mu\,, \tag{1.4}$$

for $X \in \mathrm{Vec}_c(\mathbb{R}^{n+1})$. Together with (1.2) this yields the first variation formula

$$\frac{\partial}{\partial t}\bigg|_{t=0} \int_{\phi_t(M)} 1\, d\mu_t = \int_M \mathrm{div}^M(X)\, d\mu = -\int_M \langle X, \vec{H}\rangle_e\, d\mu\,. \tag{1.5}$$

We recall the *Laplace-Beltrami operator* on functions $f : M \to \mathbb{R}$ given by

$$\Delta^M f = \mathrm{div}^M(\nabla^M f)\ .$$

We write simply Δ instead of Δ^M. One can easily check that

$$\Delta^M f = g^{ij}\nabla_i\nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}\right) = \frac{1}{\sqrt{\det g_{ij}}}\frac{\partial}{\partial x_i}\left(\sqrt{\det g_{ij}}\, g^{ij}\frac{\partial f}{\partial x_j}\right)\ .$$

The divergence theorem then gives the usual integration by parts formula

$$\int_M f\Delta h\, d\mu = -\int_M \langle \nabla f, \nabla h\rangle\, d\mu = \int_M h\Delta f\, d\mu\ .$$

If f is a function on the ambient space we have by the above calculations

$$\begin{aligned}\Delta^M f &= \operatorname{div}^M(\nabla^M f) = \operatorname{div}^M(Df) - \operatorname{div}^M(Df^{\perp})\\ &= \Delta^{\mathbb{R}^{n+1}} f - D^2 f(\nu,\nu) + \langle Df, \vec{H}\rangle_e\ .\end{aligned} \tag{1.6}$$

Thus Δ^M not only neglects the contribution of the second derivatives normal to M, but also takes into account the curvature of M.

Let $X = (x_1, \ldots, x_{n+1})$ be the coordinates of $\mathbb{R}^{n+1}$. Equation (1.6) yields

$$\Delta^M x_i = \langle \vec{H}, e_i\rangle_e$$

where e_i is the i-th basis vector of $\mathbb{R}^{n+1}$. We can thus write

$$\Delta^M X = \vec{H}\ .$$

Note that in coordinates the vectorfield X is just given by F, and we can write

$$\Delta^M F = \vec{H}\ .$$

We also note the identity

$$\Delta^M |X|_e^2 = 2n + 2\langle X, \vec{H}\rangle_e\ . \tag{1.7}$$

The second fundamental form corresponds in a certain sense to second derivatives of an immersion, and its symmetry reflects that second partial derivatives of a function commute. Similarly the Codazzi equations can be seen as a geometric manifestation that third partial derivatives commute. Thus we can also expect that there is a symmetry of the second covariant derivatives of the second fundamental form. This identity is known as *Simon's identity*:

$$\nabla_k\nabla_l h_{ij} = \nabla_i\nabla_j h_{kl} + h_{kl}h_i^{\ m}h_{mj} - h_{km}h_{il}h_j^m + h_{kj}h_i^{\ m}h_{ml} - h_k^{\ m}h_{ij}h_{ml} \tag{1.8}$$

For a proof see [27]. We note the following two consequences

$$\Delta h_{ij} = \nabla_i\nabla_j H + H h_i^{\ m}h_{mj} - h_{ij}|A|^2 \tag{1.9}$$

and

$$\frac{1}{2}\Delta|A|^2 = h^{ij}\nabla_i\nabla_j H + |\nabla A|^2 + H\operatorname{tr}(A^3) - |A|^4\ . \tag{1.10}$$

We give the explicit expressions of the main geometric quantities in the case when $F(M)$ is the graph of a function $x_{n+1} = u(x_1, \ldots, x_n)$. We choose the orientation where ν points downwards. By straightforward computations one gets

$$\nu = \frac{(D_1 u, \ldots, D_n u, -1)}{\sqrt{1+|Du|^2}}\,, \tag{1.11}$$

$$g_{ij} = \delta_{ij} + D_i u D_j u, \qquad g^{ij} = \delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\,, \tag{1.12}$$

$$h_{ij} = \frac{D^2_{ij} u}{\sqrt{1+|Du|^2}}, \qquad H = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\,, \tag{1.13}$$

where div is the standard divergence on $\mathbb{R}^n$.

1.2 Maximum principles

We will need the following maximum principles. The first one is the standard maximum principle for scalar functions:

Theorem 1.2.1 (Strong maximum principle for parabolic equations)**.**
Let M be closed and $f : M \times [0,T) \to \mathbb{R}$ satisfy

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + c\, f$$

for some smooth funtions b^i, c, where $c \geq 0$. If $f(\cdot,0) \geq 0$ then

$$\min_M f(\cdot,t) \geq \min_M f(\cdot,0)\,.$$

Furthermore, if $f(p,t_0) = \min_M f(\cdot,0)$ for some $p \in M$, $t > 0$, then $f \equiv \min_M f(\cdot,0)$ for $0 \leq t \leq t_0$.

For a proof see for example [15, Chapter 6.4 and Chapter 7.1.4] . The maximum principle can be extended to symmetric 2-tensors:

Theorem 1.2.2 (Strong parabolic maximum principle for symmetric 2-tensors (Hamilton))**.** *Let M be closed and $m^i_{\ j}$ be a symmetric bilinear form, which solves*

$$\frac{\partial m^i_{\ j}}{\partial t} \geq \Delta m^i_{\ j} + \phi^i_{\ j}(m^i_{\ j})\,,$$

where $\phi^i_{\ j}$ is a symmetric bilinear form, depending on $m^i_{\ j}$, with the property

$\phi^i_j(m^i_j) \geq 0$ if $m^i_j \geq 0$. If $m^i_j \geq 0$ for $t = 0$ then $m^i_j \geq 0$ for all $t \geq 0$. Furthermore, for $t > 0$, the rank of the null-space of m^i_j is constant, and the null-space is invariant under parallel transport and invariant in time.

For a proof see [18, Lemma 8.2]. It is helpful to think about m^i_j being in diagonal form and applying the parabolic scalar maximum principle to the smallest eigenvalue (there is actually a way to prove the maximum principle using this idea - one needs to find a way how to approximate the minimum of n functions in a smooth way preserving convexity).

We also note the strong elliptic maximum principle:

Theorem 1.2.3 (Strong elliptic maximum principle)**.** *Let M be closed and $f : M \to \mathbb{R}$ satisfy*

$$-\Delta f + b^i \nabla_i f + c\, f \leq 0$$

for some smooth funtions b^i, c, where $c \geq 0$. If $f \leq 0$, but $f \neq 0$, then $f < 0$.

For a proof see [15, §6.4, Theorem 4].

2 Basic properties

Let M^n be closed (or non-compact and complete), and $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth family of immersions. Let $M_t := F(M,t)$. We call this family a mean curvature flow starting at an initial immersion F_0, if

$$\begin{aligned} \frac{\partial F}{\partial t} &= -H \cdot \nu = \vec{H} \quad (= \Delta_{M_t} F) \\ F(\cdot, 0) &= F_0 \ . \end{aligned} \tag{2.1}$$

Remark 2.0.1: i) In general, it suffices to ask that

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} \ .$$

One solves the ODE on M given by

$$\frac{\partial \phi}{\partial t} = -dF^{-1}\left(\left(\frac{\partial F}{\partial t}\right)^{T}\right)(\phi)$$

with $\phi(0) = \mathrm{id}$. Then $\tilde{F} := F \circ \phi$ solves usual MCF.

ii) The evolution equation for a surface, which is locally given as the graph of a function u, is thus

$$\left(\frac{\partial u}{\partial t}\, e_{n+1}\right)^{\perp} = \vec{H}$$

or equivalently

$$\frac{\partial u}{\partial t}\, \langle e_{n+1}, \nu \rangle = -H \ ,$$

which yields

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\, \operatorname{div}\!\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1 + |Du|^2}\right) D_i D_j u \, . \tag{2.2}$$

This is a quasilinear parabolic equation.

iii) By formula (1.4) we have for an evolution with normal speed $-f\nu$ that

$$\frac{d}{dt}|M_t| = \frac{d}{dt}\int_M 1\, d\mu_t = -\int_M fH\, d\mu\,,$$

and thus by the Hölder's inequality, mean curvature flow decreases area the fastest, when comparing with speeds with the same L^2-norm.

Examples: There are not many explicit examples of mean curvature flow solutions.

i) The most basic one is the evolution of a sphere with initial radius $R > 0$. Assuming that the solutions remains rotationally symmetric (which follows from uniqueness, see later), we obtain the following ODE for the radius $r(t)$:

$$\frac{\partial r}{\partial t} = -\frac{n}{r}.$$

with initial condition $r(0) = R$. Integrating yields $r(t) = \sqrt{R^2 - 2nt}$. Note that the maximal existence time $T = R^2/(2n)$ is finite and the curvature blows up for $t \to T$. Furthermore, the shrinking sphere is an example of a solution which only moves by scaling, a so-called *self-similar shrinker.*

By the previous example the evolution of a cylinder

$$\mathbb{S}^k_R \times \mathbb{R}^{n-k}$$

remains cylindrical with radius given by $r(t) = \sqrt{R^2 - 2kt}$. Note that again this solution is *self-similarly shrinking.*

Another class of examples are translating solutions. Assuming that they translate with speed one in direction τ, they satisfy the elliptic equation

$$H = -\langle \tau, \nu \rangle.$$

Assuming that the solution is graphical, i.e. $x_{n+1} = u(x_1, \cdots, x_n)$, and moving in e_{n+1} direction we obtain from (2.2) that it satisfies the equation

$$\left(\delta^{ij} - \frac{D^i u\, D^j u}{1 + |Du|^2}\right) D_i D_j u = 1\,.$$

In one dimension the equation becomes

$$y_{xx} = 1 + y_x^2$$

which can be integrated explicitly, yielding $y(x) = -\ln\cos x$ for $|x| < \pi/2$, up to translation and adding constants. This solution is usually called the *grim reaper*. In higher dimensions it can be shown that there is a unique, convex, rotationally symmetric solution - but which is defined on the whole space. For properties of this solution see [8]. For $n = 2$ these are the unique convex translating entire graphs, but for $n \geq 3$ there exist entire convex translating graphs which are not rotationally symmetric, see [37].

The upwards translating grim reaper given by $e^{-y(t)} = e^{-t}\cos x(t)$ and the downwards translating grim reaper given by $e^{y(t)} = e^{-t}\cos x(t)$ can be combined to give another pair of solutions given implicitly as the solution set of

$$\cosh y(t) = e^t \cos x(t)\ , \tag{2.3}$$

and

$$\sinh y(t) = e^t \cos x(t)\ . \tag{2.4}$$

The *paperclip*, given as solution of (2.3) restricted to $|x| < \pi/2$ desribes a compact *ancient* solution that for $t \to 0$ becomes extinct in a round point and for $t \to -\infty$ looks like two copies of the grim reaper glued together smoothly. The *hairclip* (2.4) is an *eternal* solution, which for $t \to -\infty$ looks like an infinite row of grim reapers, alternating between translating up and translating down, and for $t \to +\infty$ converges to a horizontal line.

We have the following short-time existence result.

Theorem 2.0.2 (Short-time existence). *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth im-*

mersion of a closed n-dimensional manifold M. Then there exists a unique smooth solution on a maximal time interval $[0, T)$ for $T \in (0, \infty]$.

The difficulty to prove this result comes from the geometric nature of the flow, which makes any solution invariant under diffeomorphisms of M and thus the evolution equation is only weakly parabolic. There different ways to prove this result. One can either follow the approach of Hamilton [] for the Ricci flow and use the Nash-Moser Implicit function theorem. Alternatively one can use the so-called De Turck to break the diffeomeorphism invariance. The most natural way is to write the evolving surfaces $M_t = F(M, t)$ for a short time as an exponential normal graph over $M_0 = F_0(M)$. One can then check that the height function u satisfies a quasilinear parabolic equation similar to (2.2) for which standard results for those type of equations can be applied. For details see [27].

The strong maximum principle implies the following.

Theorem 2.0.3 (Avoidance principle)**.** *Assume two solutions to mean curvature flow $(M_t^1)_{t\in[0,T)}$ and $(M_t^2)_{t\in[0,T)}$ are initially disjoint (and at least one of them is compact), i.e. $M_0^1 \cap M_0^2 = \emptyset$. Then $M_t^1 \cap M_t^2 = \emptyset \;\; \forall\, t \in (0, T)$.*

Proof. Assume that this is not the case. Then there exists a first time $t_0 \in (0, T)$ where $M_{t_0}^1$ and $M_{t_0}^2$ touch at the point $x_0 \in \mathbb{R}^{n+1}$. Note that this implies that $T_{x_0} M_{t_1}^1 = T_{x_0} M_{t_1}^2 := T$ and there is an $\varepsilon > 0$ such that we can write $(M_t^1)_{t_0-\varepsilon\le t\le t_0}$ and $(M_t^2)_{t_0-\varepsilon\le t\le t_0}$ locally as graphs over the affine space $x_0 + T$. The two graph functions u_1, u_2 satisfy (2.2) which we write as

$$\frac{\partial u}{\partial t} = \left(\delta^{ij} - \frac{D^i u\, D^j u}{1 + |Du|^2}\right) D_{ij} u =: a^{ij}(Du) D_{ij} u.$$

We can assume w.l.o.g that $u_2 \le u_1$ and $u_1 = u_2$ at (x_0, t_0). But note that

$v = u_1 - u_2$ satisfies a linear parabolic equation:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= a^{ij}(Du_1)D_iD_ju_1 - a^{ij}(Du_2)D_iD_ju_2 \\
&= \int_0^1 \frac{d}{ds}\big(a^{ij}(D(su_1 + (1-s)u_2)D_{ij}(su_1 + (1-s)u_2)\big)\, ds \\
&= \bigg(\int_0^1 a^{ij}(D(su_1 + (1-s)u_2))\, ds\bigg) D_{ij}v \\
&\quad + \bigg(\int_0^1 \frac{\partial a^{ij}}{\partial p_k}(D(su_1 + (1-s)u_2))D_{ij}(su_1 + (1-s)u_2)\, ds\bigg) D^k v \\
&=: \tilde{a}^{ij} D_{ij} v + \tilde{b}^k D_k v\,,
\end{aligned}$$

where p is the Du variable of $a^{ij}(p)$. Note that $\tilde{a}^{ij}$ is symmetric and strictly positve. Since $v \geq 0$ and $v = 0$ at (x_0, t_0) the strong maximum principle implies that $v \equiv 0$ which yields a contradiction. □

With more or less the same argument one can show the following.

Corollary 2.0.4 (Preservation of embeddedness)**.** *If M_0 is closed and embedded, then M_t is embedded for all t.*

Remark 2.0.5: (i) Enclosing a compact initial hypersurface M_0 by a large sphere, and using that the maximal existence time of the evolution of the sphere is finite, we obtain that the maximal existence time T is finite.

(ii) Note the we can translate a solution to mean curvature flow in the ambient space and get a new solution to mean curvature flow. Thus the avoidance principle implies that the distance between two disjoint solutions is non-decreasing in time.

(iii) In case M_0 is embedded, we will always choose ν to be the *outward* unit normal.

2.0.1 Outline of the course

First, we will compute the evolution equations of the main geometric quantities and show for example that convexity and non-negative mean curvature are preserved. Then we will show that the flow exists smoothly as long as the second fundamental form stays bounded.

A main tool in the analysis of singularities is Huisken's monotonicity formula. We will derive it, and show that it implies that any tangent flow (if it exists) is a self-similarly shrinking solution. Following an argument of White [40], we will use the monotonicity formula to show that a control on the Gaussian density ratios implies a control on the curvature. We will conclude with the classification of mean convex self-similarly shrinking solutions and self-similarly shrinking curves in the plane.

For mean curvature flow of curves in the plane, the so-called *curve shortening flow*, the following theorem holds:

Theorem 2.0.6 (Gage/Hamilton [16], Grayson [17])**.** *Under curve shortening flow, simple, closed curves become convex in finite time and shrink to a 'round' point.*

We will not follow the original proof, but use Huisken's monotonicity formula and a quantitative control of embeddedness, which will rule out certain singularities.

In higher dimensions one cannot expect that such a behaviour is true, since one can rather easily construct counterexamples. But the following fundamental result of Huisken holds:

Theorem 2.0.7 (Huisken [24])**.** *Any closed, convex hypersurface becomes immediately strictly convex under mean curvature flow and converges in finite time to a 'round' point.*

We will give a proof of this result, making again strong use of the monotonicity formula.

The next part will focus on two-convex mean curvature flow, that is when $\lambda_1+\lambda_2 \geq 0$ everywhere on M_0, which we will see is preserved under the evolution. We will show that this implies that the only possible singularities are asymptotic either to a shrinking sphere or a shrinking cylinder with only *one* straight direction. We will then present the result of Huisken-Sinestrari that this structure allows one to define a mean curvature flow with surgery:

Theorem 2.0.8 (Huisken-Sinestrari [30]). *Let $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed n-dimensional hypersurface with $n \geq 3$. Assume M_0 is two-convex. Then there exists a mean curvature flow with surgeries starting from M_0 which terminates after a finite number of steps.*

The result has topological consequences, which we will also discuss. It is important to note that this is the extrinsic analogue of the results of Hamilton/Perelman on 3-dimensional Ricci flow with surgeries / through singularities.

Furthermore, we will also discuss the recent result Brendle [7] that the only embedded $M^2 \subset \mathbb{R}^3$ which is self-shrinking and diffeomorphic to $\mathbb{S}^2$ is round. If time permits, we are planning to discuss applications of mean curvature flow to prove optimal isoperimetric inequalities.

Here is a list of further introductory texts on mean curvature flow (which I have partially used and copied from in preparation of these notes):

- B. White, *Topics in mean curvature flow*, lecture notes by O. Chodosh. Available at https://web.math.princeton.edu/~ochodosh/notes.html
- K. Ecker, *Regularity theory for Mean Curvature Flow*, Birkhäuser
- M. Ritoré and C. Sinestrari, *Mean Curvature Flow and isoperimetric inequalities*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser
- C. Mantegazza, *Lecture Notes in Mean Curvature Flow*, Progress in Mathematics, Volume 290, Birkhäuser
- R. Haslhofer, *Lectures on curve shortening flow.* Available at http://www.math.toronto.edu/roberth/pde2/curve_shortening_flow.pdf
- R. Haslhofer, *Lectures on mean curvature flow.* Available at https://

arxiv.org/abs/1406.7765.

2.1 The maximal time of existence

We first compute the basic evolution equations.

Lemma 2.1.1. *The following evolution equations hold.*

(i) $\frac{\partial}{\partial t}\nu = \nabla H$

(ii) $\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij}$

(iii) $\frac{\partial}{\partial t}g^{ij} = 2Hh^{ij}$

(iv) $\frac{\partial}{\partial t}d\mu = -H^2\,d\mu$

(v) $\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{im}h^m_{\ j} + |A|^2 h_{ij}$

(vi) $\frac{\partial}{\partial t}h^i_{\ j} = \Delta h^i_{\ j} + |A|^2 h^i_{\ j}$

(vii) $\frac{\partial}{\partial t}H = \Delta H + |A|^2 H$

(viii) $\frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4$

Proof. (i) We first note that $\langle \nu, \nu\rangle \equiv 1$ so we obtain

$$\Big\langle \frac{\partial \nu}{\partial t}, \nu \Big\rangle = 0\,.$$

Since $\langle \nu, \frac{\partial F}{\partial x_i}\rangle \equiv 1$ we can compute

$$\Big\langle \frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x_i}\Big\rangle = -\Big\langle \nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_i}\Big\rangle = \Big\langle \nu, \frac{\partial}{\partial x_i}(H\nu)\Big\rangle = \frac{\partial H}{\partial x_i}\ ,$$

where we used that $\langle \frac{\partial}{\partial x_i}\nu, \nu\rangle = 0$. This yields the statement.

(ii) We have

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\Big\rangle = -\Big\langle \frac{\partial}{\partial x_i}(H\nu), \frac{\partial F}{\partial x_j}\Big\rangle - \Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(H\nu)\Big\rangle \\ &= -Hh_{ij}\end{aligned}$$

(iii) This follows from differentiating the identity

$$g^{il}g_{lj} = \delta^i_{\ j}\,.$$

(iv) This follows since by (1.2) and following calculation we have

$$\frac{\partial}{\partial t} d\mu = \mathrm{div}^M(\vec{H})\, d\mu = -\langle \vec{H}, \vec{H}\rangle\, d\mu = -H^2\, d\mu\,.$$

(v) We choose normal coordinates at (p,t). Note that this implies that all Christoffel symbols at that point vanish and the partial derivatives coincide with the covariant derivatives.

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij} &= \frac{\partial}{\partial t}\Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial x_j}\Big\rangle = -\Big\langle \frac{\partial}{\partial x_i}(H\nu), \frac{\partial \nu}{\partial x_j}\Big\rangle + \Big\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(\nabla H)\Big\rangle \\ &= -H h_{im} h^m_j + \nabla_j \nabla_i H\end{aligned}$$

Combining this with Simon's identity (1.9) yields

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{im} h^m_j + h_{ij}|A|^2$$

(vi) Follows from (v) combined with (iii).

(vii) Follows from (vi) by taking a trace.

(viii) Follows from (vi) by writing $|A|^2 = h^i{}_j h_i{}^j$ and noting that in normal coordinates at a point (p,t)

$$\Delta |A|^2 = \sum_l \nabla_l \nabla_l h^i{}_j h_i{}^j = h^i{}_j \Delta h_i{}^j + h_i{}^j \Delta h^i{}_j + 2|\nabla A|^2\ .$$

□

By the strong maximum principle we obtain the following two theorems.

Theorem 2.1.2. *Assume $M_0 = F_0(M)$ closed and mean convex, i.e. $H \geq 0$. Then $H > 0$ for all $t > 0$.*

Proof. That $H \geq 0$ for $t \geq 0$ follows from the evolution equation of H and the parabolic maximum principle, Theorem 1.2.1. Assume now that $H(p_0,t_0) = 0$ for some $t_0 > 0$. The strong maximum principle then implies that $H \equiv 0$ for all (p,t) and $0 \leq t \leq t_0$. But this is impossible since any closed hypersurface in $\mathbb{R}^{n+1}$ has points where $\lambda_1 > 0$. □

Theorem 2.1.3. *Assume $M_0 = F_0(M)$ closed and convex, i.e. ${h^i}_j \geq 0$. Then ${h^i}_j > 0$ for all $t > 0$.*

Proof. That ${h^i}_j \geq 0$ for $t \geq 0$ follows from the evolution equation of ${h^i}_j$ and the parabolic maximum principle for 2-tensors, Theorem 1.2.2. Assume now that ${h^i}_j(p_0, t_0)$ has a zero eigenvalue for some $t_0 > 0$. The strong maximum principle then implies that the rank of the null-space is greater or equal to one for all (p, t) and $0 \leq t \leq t_0$. But this again is impossible since there exist points where $\lambda_1 > 0$. □

We now aim to show that the solution exists as long as $|A|$ stays bounded. To do this we first need the evolution equation of higher covariant derivatives of A. We will use the notation $S*T$ to denote any linear combination formed by contraction on S and T by g.

Lemma 2.1.4.

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A$$

Proof. We note that the Christoffel symbols are not tensorial, but the difference of Christoffel symbols is, and thus also their time derivative. We can thus compute at a point p in normal coordinates: Γ^i_{jk} is given by

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^i_{jk} &= \frac{\partial}{\partial t}\left(\frac{1}{2}g^{il}\left(\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}\right)\right) \\ &= \frac{1}{2}g^{il}\left(\frac{\partial}{\partial x_j}\frac{\partial g_{kl}}{\partial t} + \frac{\partial}{\partial x_k}\frac{\partial g_{jl}}{\partial t} - \frac{\partial}{\partial x_l}\frac{\partial g_{jk}}{\partial t}\right) \\ &= -g^{il}\left(\frac{\partial}{\partial x_j}(Hh_{kl}) + \frac{\partial}{\partial x_k}(Hh_{jl}) - \frac{\partial}{\partial x_l}(Hh_{jk})\right) = A * \nabla A\end{aligned} \tag{2.5}$$

Claim:

$$\frac{\partial}{\partial t}(\nabla^m h_{ij}) = \Delta(\nabla^m h_{ij}) + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A\ . \tag{2.6}$$

The claim is true for $m = 0$. We argue by induction, using (2.5)

$$\begin{aligned}
\frac{\partial}{\partial t}(\nabla^{m+1} h_{ij}) &= \nabla \frac{\partial}{\partial t}(\nabla^m h_{ij}) + A * \nabla A * \nabla^m A \\
&= \nabla\Big(\Delta(\nabla^m h_{ij}) + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A\Big) \\
&= \Delta(\nabla^{m+1} h_{ij}) + A * A * \nabla^{m+1} A + \sum_{i+j+k=m+1} \nabla^i A * \nabla^j A * \nabla^k A \\
&= \Delta(\nabla^{m+1} h_{ij}) + \sum_{i+j+k=m+1} \nabla^i A * \nabla^j A * \nabla^k A \, .
\end{aligned}$$

where we used the Gauss equations in the second last line to express $R_{ijkl} = A * A$ which appears when interchanging covariant derivatives. This proves (2.6). The lemma then follows since

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = 2\langle \nabla^m A, \frac{\partial}{\partial t}\nabla^m A\rangle + A * A * \nabla^m A * \nabla^m A$$

and

$$\Delta|\nabla^m A|^2 = 2\langle \nabla^m A, \Delta\nabla^m A\rangle + 2|\nabla^{m+1} A|^2 \, .$$

□

With this we can show that all higher derivatives of A stay bounded if A is bounded.

Proposition 2.1.5. *If $|A|^2 \leq C_0$ on $M \times [0, T)$, then*

$$|\nabla^m A|^2 \leq C_m \qquad \textit{on } M \times [0, T) \, ,$$

where $C_m = C_m(n, M_0, C_0)$.

Proof. We have

$$\frac{\partial}{\partial t}|\nabla^m A|^2 \leq \Delta|\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + C(n, m) \sum_{i+j+k=m} |\nabla^i A| \cdot |\nabla^j A| \cdot |\nabla^k A| \cdot |\nabla^m A| \, .$$

We give a proof by induction. The case $m = 0$ is trivially true. So we assume that for $m > 0$ we have $|\nabla^l A|^2 \leq C_l$ for $0 \leq l \leq m-1$. Thus

$$\frac{\partial}{\partial t}|\nabla^{m-1}A|^2 \leq \Delta|\nabla^{m-1}A|^2 - 2|\nabla^m A|^2 + B_{m-1}$$
$$\frac{\partial}{\partial t}|\nabla^m A|^2 \leq \Delta|\nabla^m A|^2 + B_m(1 + |\nabla^m A|^2)\,.$$

We consider the function $f := |\nabla^m A|^2 + B_m|\nabla^{m-1}A|^2$, which satisfies

$$\frac{\partial f}{\partial t} \leq \Delta f - B_m|\nabla^m A|^2 + B \leq \Delta f - B_m f + B'\ .$$

Thus $\tilde{f} := e^{B_m t} f - B't$ satisfies

$$\frac{\partial \tilde{f}}{\partial t} \leq \Delta \tilde{f}$$

which implies $\tilde{f}(t) \leq \max_M \tilde{f}(0)$ and thus

$$f(t) \leq e^{-B_m t}(\max_M \tilde{f}(0) + tB') \leq C'$$

□

Let us assume from now on that $[0, T)$ is the maximal time of existence of the flow.

Corollary 2.1.6. *We have* $\limsup_{t\to T} \max_{M_t} |A|^2 = \infty$.

Proof. Let us assume to the contrary that $|A|^2 \leq C_0$ for $t \in [0, T)$. By Proposition 2.1.5 all higher derivatives of A are bounded. This implies that $F(\cdot, t)$ converges smoothly to a limiting immersion $F(\cdot, T)$, see the exercise below . But by short-time existence this implies that we can extend the solution further, which contradicts the assumption that T is maximal. □

Exercise 2.1.7: (i) Assume

$$F_i : M \to \mathbb{R}^{n+1}$$

is a sequence of immersions of a closed n-dimensional manifold M such that $F_i(M) \subset B_R(0)$ for some $R > 0$ and all i. Furthermore, assume that there exists numbers $C_m < \infty$ such that

$$\sup_{M,i} |\nabla^m A_{F_i}| \le C_m$$

for all $0 \le m < \infty$ and there exists $\Lambda > 0$ such that

$$\Lambda^{-1} g_p^0(\xi,\xi) \le g_p^i(\xi,\xi) \le \Lambda g_p^0(\xi,\xi)$$

for all $i \in \mathbb{N}$, all $p \in M$ and all $\xi \in T_pM$. Show that there exists a subsequence such that F_i converges to a limiting immersion F^∞.

(ii) Use the evolution equation of the metric and (i) to complete the proof of Corollary 2.1.6.

One can even show that bounds on the second fundamental form imply local bounds for all higher derivatives.

Theorem 2.1.8 (Ecker/Huisken [13, 12])**.** *Let (M_t) be a smooth, properly embedded solution of mean curvature flow in $B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ which satisfies the estimate*

$$|A(x)|^2 \le \frac{C_0}{\rho^2}$$

for all $x \in M_t \cap B_\rho(x_0)$ and $t \in (t_0 - \rho^2, t_0)$. Then for every $m \ge 1$ there is a constant C_m, depending only on n, m and C_0 such that for all $x \in M_t \cap B_{\rho/2}(x_0)$ and $t \in (t_0 - \rho^2/4, t_0)$,

$$|\nabla^m A|^2 \le \frac{C_m}{\rho^{2(m+1)}}\ .$$

2.2 The monotonicity formula

In this section we will discuss Huisken's monotonicity formula, White's local regularity theorem and the classification of self-shrinkers for non-negative mean curvature and for curves in the plane. Let $\mathcal{M} = \{M_t \subset \mathbb{R}^{n+1}\}$ be a smooth mean curvature flow of hypersurfaces with at most polynomial volume growth. Let

$X_0 = (x_0, t_0)$ be a point in space time, and consider

$$\rho_{X_0}(x,t) := (4\pi(t_0 - t))^{-n/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} ,$$

which is the backward heat kernel in $\mathbb{R}^{n+1}$, based at (x_0, t_0) and scaled by a factor $(4\pi(t_0 - t))^{1/2}$.

Theorem 2.2.1 (Huisken's monotonicity formula [25])**.**

$$\frac{d}{dt} \int\limits_{M_t} \rho_{X_0} \, d\mu = - \int\limits_{M_t} \left| \vec{H} + \frac{(x - x_0)^{\perp}}{2(t_0 - t)} \right|^2 \rho_{X_0} \, d\mu \qquad (t < t_0) \, .$$

One way to interpret this formula is as a weighted version of the monotonicity of the area, see Lemma 2.1.1 (iv). However, note the following scaling invariance.

Exercise 2.2.2 (Parabolic Rescaling)**:** (i) Let $\lambda > 0, x' = \lambda(x - x_0), t' = \lambda^2(t - t_0)$ and consider the rescaled flow

$$M^\lambda_{t'} = \lambda(M_{t_0 + \lambda^{-2} t'} - x_0) \; .$$

Show that this is again a mean curvature flow.

(ii) Show that

$$\int\limits_{M_t} \rho_{X_0}(x,t) \, d\mu_t(x) = \int\limits_{M^\lambda_{t'}} \rho_0(x', t') \, d\mu_{t'}(x') \qquad (t' < 0) \; .$$

Exercise 2.2.3 (Self-similar shrinkers)**:** Let $\{M_t \subset \mathbb{R}^{n+1}\}_{t \in (\infty, 0)}$ be an ancient solution of mean curvature flow. Show that $\vec{H} - \frac{x^\perp}{2t} = 0$ for all $t < 0$ if and only if $M_t = \sqrt{-t} M_{-1}$ for all $t < 0$.

Proof of Theorem 2.2.1. We can assume $X_0 = (0,0)$ and we write $\rho = \rho_0$. Recall that by (1.6) we have that

$$\Delta \rho = \Delta^{\mathbb{R}^{n+1}} \rho - D^2\rho(\nu, \nu) + \langle D\rho, \vec{H} \rangle \; .$$

Since $\frac{d}{dt}\rho = \frac{\partial}{\partial t}\rho + \langle D\rho, \vec{H}\rangle$ we have

$$\begin{aligned}
\frac{d}{dt}\rho + \Delta\rho &= \frac{\partial}{\partial t}\rho + \Delta^{\mathbb{R}^{n+1}}\rho - D^2\rho(\nu,\nu) + 2\langle D\rho, \vec{H}\rangle \\
&= \frac{\partial}{\partial t}\rho + \Delta^{\mathbb{R}^{n+1}}\rho - D^2\rho(\nu,\nu) + \frac{|D^\perp\rho|^2}{\rho} - \left|\vec{H} - \frac{D^\perp\rho}{\rho}\right|^2 + H^2\rho
\end{aligned}$$

One can check directly that $\frac{\partial}{\partial t}\rho + \Delta^{\mathbb{R}^{n+1}}\rho - D^2\rho(\nu,\nu) + \frac{|D^\perp\rho|^2}{\rho} = 0$ and thus

$$\frac{d}{dt}\rho + \Delta\rho - H^2\rho = -\left|\vec{H} - \frac{x^\perp}{2t}\right|^2\rho\ .$$

Together with the evolution equation for the measure this yields

$$\frac{d}{dt}\int_{M_t}\rho\, d\mu = -\int_{M_t}\left|\vec{H} - \frac{x^\perp}{2t}\right|^2\rho\, d\mu \qquad (t<0)\,.$$

□

Exercise 2.2.4 (Local version [12])**:** If M_t is only defined locally, say in $B(x_0,\sqrt{4n}\rho)\times(t_0-\rho^2,t_0)$, then we can use the cutoff function $\varphi^\rho_{X_0}(x,t) = (1-\frac{|x-x_0|^2+2n(t-t_0)}{\rho^2})^3_+$. Show that $(\frac{d}{dt}-\Delta)\varphi^\rho_{X_0}\le 0$ and thus we still get the monotonicity inequality

$$\frac{d}{dt}\int_{M_t}\varphi^\rho_{X_0}\rho_{X_0}\, d\mu \le -\int_{M_t}\left|\vec{H} - \frac{(x-x_0)^\perp}{2(t-t_0)}\right|^2\rho_{x_0}\varphi^\rho_{X_0}\, d\mu\,.$$

We define the Gaussian density ratios of the flow $\mathcal{M} = \{M_t\}$ with respect to $X=(x_0,t_0)$ as

$$\Theta(\mathcal{M},X,r) = \int_{M_{t_0-r^2}}\rho_X\, d\mu\,.$$

Note that the monotonicity formula implies that $\Theta(\mathcal{M},X,r)$ is increasing in r. In case the flow is only defined locally as in Remark 2.2.4 we set

$$\Theta^\rho(\mathcal{M},X,r) = \int_{M_{t_0-r^2}}\varphi^\rho_{X_0}\ \rho_X\, d\mu\,.$$

Hence as $r \searrow 0$, the limit exists, so we can set

$$\Theta(\mathcal{M}, X) := \lim_{r \searrow 0} \Theta(\mathcal{M}, X, r)\,,$$

called the *Gaussian density* of $\mathcal{M}$ at X.

Suppose f is a continuous, bounded (or more generally $|f| \leq C(1+|x|)^k$), and assume $M^i \to M$ locally smoothly and the M^i have uniform area volume growth. Then is easy to see that (using a cutoff function if necessary)

$$\int_{M^i} f e^{-\frac{|x|^2}{4r^2}}\, d\mu_i \to \int_M f e^{-\frac{|x|^2}{4r^2}}\, d\mu\,.$$

Proposition 2.2.5. *Assume $\mathcal{M}_i \to \mathcal{M}$ locally smoothly, $X_i \to X, r_i \to 0$. Then*

$$\limsup_i \Theta(\mathcal{M}^i, X_i) \leq \limsup_i \Theta(\mathcal{M}^i, X_i, r_i) \leq \Theta(\mathcal{M}, X)\,.$$

Proof. Translating by X_i, we can assume $X_i = X = 0$. Then, for $r > 0$ and for i sufficiently large, we have $r_i < r$. Thus

$$\limsup_i \Theta(\mathcal{M}_i, 0) \leq \limsup_i \Theta(\mathcal{M}_i, 0, r_i) \leq \limsup_i \Theta(\mathcal{M}_i, 0, r) = \Theta(\mathcal{M}, 0, r)\ .$$

This holds for all $r > 0$. Letting $r \searrow 0$, the proposition follows. $\square$

We will see that the monotonicity formula implies that close to a singularity at $X = (x_0, t_0)$ a mean curvature flow is nearly self-similar - that is it is nearly moving only by homotheties. Consider, as in Exercise 2.2.2 (i) the rescaled flow

$$M^\lambda_{t'} = \lambda(M_{t_0+\lambda^{-2}t'} - x_0)\ . \tag{2.7}$$

By Exercise 2.2.2 (i) we have for any $r > 0$

$$\begin{aligned} \Theta(\mathcal{M}, X, \lambda^{-1}r) - \Theta(\mathcal{M}, X) &= \Theta(\mathcal{M}^\lambda, 0, r) - \Theta(\mathcal{M}, X) \\ &= \int_{-r^2}^{0} \int_{M^\lambda_t} \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 \rho_0\, d\mu\, dt \end{aligned} \tag{2.8}$$

We now consider a sequence $\lambda_i \to \infty$ and we assume that $(M_t^{\lambda_i})$ converges smoothly to a limiting mean curvature flow $(\mathcal{M}_\infty)$, defined on $(-\infty, 0)$, then the above formula implies that M_t^∞ satisfies

$$\vec{H} - \frac{x^\perp}{2t} = 0 \,.$$

for $t < 0$.

Exercise 2.2.6: One calls a singularity at time T of type I, if one has the bound

$$\sup_{M_t} |A|^2 \leq \frac{C}{T-t}$$

for some C. Let $t_0 = T$. Doing a parabolic rescaling around a point (x_0, T) as in (2.7) show that this bound is scaling invariant, i.e.

$$\sup_{M^\lambda_{t'}} |A|^2 \leq \frac{C}{(-t')} \,.$$

Using the monotonocity formula show that the flows $\{M^\lambda_{t'}\}$ converge subsequentially as $\lambda \to \infty$ to a smooth limiting flow, which is self-similarly shrinking. Singularities which do not satisfy this bound are called type II singularities. Even in this case, one can still show that one can extract a weak limit, where the limiting object is not a smooth mean curvature flow anymore, but a so-called Brakke-flow. A Brakke flow is a family of moving varifolds, which satisfies mean curvature flow in an integrated sense.

Exercise 2.2.7: Let $\mathcal{M} = \{M_t\}$ be a smooth mean curvature flow. We say that $X = (x_0, t_0)$ is a smooth point of the flow, if in a space-time neighbourhood of X_0 the flow $\mathcal{M}$ is smooth. Show that at a smooth point X_0 in the support of $\mathcal{M}$ one has

$$\Theta(\mathcal{M}, X_0) = 1\,,$$

and thus at each singular point $\Theta \geq 1$. Similarly, any point reached by the flow has $\Theta \geq 1$. Assume that $\mathcal{M}$ is a smooth mean curvature flow such that X_0 is a smooth point of the flow. Show that $\Theta(\mathcal{M}, X_0, r) \equiv 1$ for all $r > 0$ if and only if $\mathcal{M}$ is a multiplicity one plane containing X_0.

We consider parabolic backwards cylinders $P((x_0,t_0),r) = B(x_0,r)\times(t_0-r^2,t_0]$.

Theorem 2.2.8 (Local regularity theorem [4, 40])**.** *There exists universal constants $\varepsilon > 0$ and $C < \infty$ with the following property: If $\mathcal{M}$ is a smooth mean curvature flow in $P(X_0, 4n\rho)$ such that*

$$\sup_{X\in P(X_0,r)} \Theta^\rho(\mathcal{M},X,r) < 1+\varepsilon$$

for some $r \in (0,\rho)$, then

$$\sup_{P(X_0,r/2)} |A| \leq Cr^{-1}\,. \tag{2.9}$$

Remark 2.2.9: (i) If $\Theta(\mathcal{M},X_0) < 1+\varepsilon/2$, then $\Theta(\mathcal{M},X,r) < 1+\varepsilon$ for all X sufficiently close to X_0 and all $r > 0$ sufficiently small.

(ii) By Theorem 2.1.8 we have

$$\sup_{P(X_0,r/4)} |\nabla^m A| \leq C_m r^{-m-1}\,.$$

Proof of Theorem 2.2.8. Suppose the assertion fails. Then there exists a sequence of smooth flows $\mathcal{M}^j$ in $P(X_0, 4n\rho_j)$, for some $\rho_j > 1$ (we can always assume via scaling that $r_j = 1$) with

$$\sup_{X\in P(0,1)} \Theta^{\rho_j}(\mathcal{M}^j,X,1) < 1+j^{-1}\ ,$$

but that there are points $X_j \in P(0,1/2)$ with $|A|(X_j) > j$.

Claim: we can find $Y_j \in P(0,3/4)$ with $Q_j = |A|(Y_j) > j$ such that

$$\sup_{P(Y_j, j/(10Q_j))} |A| \leq 2Q_j\ . \tag{2.10}$$

We do this via point selection: Fix j. If $Y_j^0 = X_j$ already satisfies (2.10) with $Q_j^0 = |A|(Y_j^0)$, we are done. Otherwise, there is a point $Y_j^1 \in P(Y_j^0, j/(10Q_j^0))$ with $Q_j^1 = |A|(Y_j^1) > 2Q_j^0$. If Y_j^1 satisfies (2.10), we are done. Otherwise there is a point $Y_j^2 \in P(Y_j^1, j/(10Q_j^1))$ with $Q_j^2 = |A|(Y_j^2) > 2Q_j^1$, etc. Note that $\frac{1}{2} + \frac{j}{10Q_j^0}(1+\frac{1}{2}+\frac{1}{4}+\ldots) < \frac{3}{4}$. By smoothness, the iteration terminates after a finite number of steps, and the last point of the iteration lies in $P(0,3/4)$ and satisfies (2.10).

Continuing the proof of the theorem, let $\hat{\mathcal{M}}^j$ be the flows obtained by shifting Y_j to the origin and parabolically rescaling by $Q_j = |A|(Y_j) \to \infty$. Since the rescaled flows satisfy $|A|(0) = 1$ and $\sup_{P(0,j/10)} |A| \leq 2$ we use Theorem 2.1.8 to pass to a smooth nonflat global limit. On the other hand, since

$$\Theta^{\hat{\rho}_j}(\hat{\mathcal{M}}^j, 0, Q_j) < 1 + j^{-1} \ ,$$

and Exercise 2.2.7, where $\hat{\rho}_j = Q_j \rho_j \to \infty$, the limit is a flat plane, a contradiction. $\square$

For $n \geq 2$ one can classify all self-similar solutions, which have non-negative mean curvature. Huisken [25] originally proved this under the assumption that $|A|$ is uniformly bounded (which is natural assumption if one has a type I singularity). For a closed mean convex mean curvature flow with $H > 0$ one can show that there exists a $C > 0$ such that the scaling invariant estimate $|A| \leq CH$ holds along the flow. Together with the equation $H = \langle x, \nu \rangle / 2$ one obtains that any smooth self-similar blow-up satisfies

$$|A| \leq CH \leq C|x| \ .$$

This is sufficient to make Huisken's proof work, where one needs to justify several integrations by parts. Colding and Minicozzi [9] removed this assumption completely. We will not discuss the proof at the moment, since we will get a similar result later with different methods.

Theorem 2.2.10 (Huisken [25], Colding/Minicozzi [9])**.** *If M^n, for $n \geq 2$, is an embedded hypersurface in $\mathbb{R}^{n+1}$, with non-negative mean curvature, satisfying $H = \langle x, \nu \rangle / 2$, then M^n is of the form*

$$\mathbb{S}^{n-m}_{(2(n-m))^{1/2}} \times \mathbb{R}^m$$

for $m = 0, \ldots, n$.

This has deep implications for the structure of singularities of mean curvature flow of mean convex surfaces. For curves in the plane no condition is needed:

Theorem 2.2.11 (Abresch/Langer [1]). *The only embedded, closed curves in* $\mathbb{R}^2$ *satisfying* $k = \langle x, \nu \rangle / 2$ *are either a straight line through the origin or the circle of radius* $\sqrt{2}$.

Exercise 2.2.12: Show that the energy

$$E := \langle X, \nu \rangle e^{-|x|^2/4}$$

is constant along any curve satisfying $k = \langle x, \nu \rangle / 2$. Use this to show that any (immersed) self-similarly shrinking solution is convex and that the only non-compact solutions are straight lines through the origin.

Remark 2.2.13: Abresch and Langer use the energy E to show that there is a countable family of closed self-similarly shrinking curves, which are uniquely characterised by their winding number w.r.t. the origin. It is rather easy to see that any solution stays in an annulus between $r_{\min} \leq \sqrt{2} \leq r_{\max}$ and the solution is periodic w.r.t. the points of maximum and minimum distance. Abresch and Langer show that the angle $\Delta\theta(r_{\min}, r_{\max})$ between these points is monotone in $r_{\min}$ to prove the above statement.

3 Evolution of closed curves in the plane

In this section we consider the evolution of closed curves in the plane under curve shortening flow, that is mean curvature flow in $\mathbb{R}^2$. The evolution equation for a smooth family of curve (γ_t) is then given by

$$\frac{d\gamma_t}{dt} = \vec{k}\,,$$

where $\vec{k}$ is the curvature vector of the curve. In the following we want to present a proof of the following two theorems.

Grayson's argument, following the work of Gage and Hamilton for convex curves, is rather delicate. More recently the proof has been simplified by using isoperimetric estimates to rule out certain types of singularities: Huisken [26] proved an estimate, controlling the ratio between the intrinsic and extrinsic distance between two points on the evolving curve, and Hamilton [20] gave an estimate controlling the ratio of the isoperimetric profile to that of a circle of the same area. The proof then follows in both cases by destinguishing type I and type II singularities. In the first case one can use Huisken's monotonicity formula and the classification of self-similarly shrinking solutions to show that the asymptotic shape of the solution is the shrinking circle. In the case of a type II singularity one can do a rescaling to produce a convex limiting curve, which by Hamilton's Harnack estimate [19] has to be the 'grim reaper' curve. But this violates the isoperimetric bound ruling out singularities of type II. Very recently, using a refined isoperimetric estimate, a very elegant and direct proof of Grayson's result has been given by Andrews and Bryan [3] which does not use Huisken's monotonicity formula or the classification of singularities. For a nice presentation, using Huiskens comparison between the intrinsic and extrinsic distance and the analysis of type II singularities, see the notes of Haslhofer [21]. They also treat most of what we have seen so far in the 1-d case.

In the following we will present Huisken's estimate on the ratio between the extrinsic and intrinsic distance. Using the monotonicity formula we then show that one can give a proof of Grayson's result by using only Huisken's monotonicity formula and the classification of self-similarly shrinking curves, thus avoiding the analysis of type II singularities.

3.1 Intrinsic versus extrinsic distance

In this section we will present Huisken's proof that given two points p, q on γ_t, the ratio between the intrinsic distance along the curve and the extrinsic distance stays controlled under curve shortening flow. We follow the original article [26]. This is one of the first examples for the use of a two point maximum principle in geometric evolution equations. This technique has recently shown to be very successful, see for example the proof of Brendle of the Lawson conjecture [5]. For an overview and a proof of the result below in a slightly more unified form see the survey of Brendle [6].

Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a closed, embedded curve moving by curve shortening flow. Let $L(t)$ be the total length of the curve, and l be the intrinsic distance between two points, which is defined for $0 \leq l \leq L/2$. Let the smooth function $\psi : \mathbb{S}^1 \times \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ be given by

$$\psi := \frac{L}{\pi} \sin\left(\frac{l\pi}{L}\right).$$

Note that since $\sin\left(\frac{l\pi}{L}\right) = \sin\left(\frac{(L-l)\pi}{L}\right)$ the function is smooth at points (p, q) with $l(p, q) = L/2$. Then the ratio d/ψ, where $d(p, q) = |F(p) - F(q)|$ is the extrinsic distance between two points on the curve, is equal to 1 on the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ for any smooth embedding of $\mathbb{S}^1 \to \mathbb{R}^2$ and the ratio d/ψ is identically

1 on any round circle.

Theorem 3.1.1 (Huisken). *The minimum of d/ψ on $\mathbb{S}^1 \times \mathbb{S}^1$ is non-decreasing under curve shortening flow.*

Proof. Since d/ψ is equal to one on the diagonal, it is sufficient to show that whenever d/ψ has a spatial minimum $(d/\psi)(p, q, t) < 1$, at some pair of points $(p, q) \in \mathbb{S}^1 \times \mathbb{S}^1$, and some time $t_0 \in [0, T)$, then

$$\frac{d}{dt}(d/\psi)(p, q, t_0) > 0\,.$$

We take s to be the arclenght parameter at time t_0, and without loss of generality $0 \leq s(p) < s(q) \leq L(t_0)/2$, such that $l(p, q) = s(q) - s(p)$. For any variation $\xi \in T_p\mathbb{S}^1_{t_0} \oplus T_q\mathbb{S}^1_{t_0}$ we have that the for the first and second variation

$$\delta(\xi)(d/\psi)(p, q, t_0) = 0 \qquad \delta^2(\xi)(d/\psi)(p, q, t_0) \geq 0$$

From the vanishing of the first variation for $\xi = e_1 \oplus 0$ and $\xi = 0 \oplus e_2$ one can easily compute that

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{\psi} \cos\left(\frac{l\pi}{L}\right), \tag{3.1}$$

where $e_1 = \frac{\partial}{\partial s}F(p, t_0)$, $e_2 = \frac{\partial}{\partial s}F(q, t_0)$ and $\omega = -d^{-1}(p, q, t_0)\big(F(p, t_0) - F(q, t_0)\big)$. This implies two possibilities.

<u>Case 1</u>: $e_1 = e_2$. Choosing $\xi = e_1 \oplus e_2$ in the second variation inequality we can compute that

$$0 \leq \delta^2(e_1 \oplus e_2)(d/\psi) = \frac{1}{\psi}\langle \omega, \vec{k}(q, t_0) - \vec{k}(p, t_0)\rangle \tag{3.2}$$

<u>Case 2</u>: $e_1 \neq e_2$. Using that in this case $e_1 + e_2$ is parallel to ω and using the second variation inequality with $\xi = e_1 \ominus e_2$ one can deduce that

$$0 \leq \delta^2(e_1 \ominus e_2)(d/\psi) = \frac{1}{\psi}\langle \omega, \vec{k}(q, t_0) - \vec{k}(p, t_0)\rangle + \frac{4\pi^2}{L^2}\frac{d}{\psi}\,. \tag{3.3}$$

We can now estimate $(d/dt)(d/\psi)$. Using the original evolution equation and that

$(d/dt)(ds) = -k^2(ds)$ we see

$$\begin{aligned}\frac{d}{dt}\Big(\frac{d}{\psi}\Big) &= \frac{1}{d\psi}\big\langle F(q,t_0) - F(p,t_0), \vec{k}(q,t_0) - \vec{k}(p,t_0)\big\rangle - \frac{d}{\psi^2}\frac{d}{dt}\Big(\frac{L}{\pi}\sin\Big(\frac{l\pi}{L}\Big)\Big) \\ &= \frac{1}{\psi}\big\langle \omega, \vec{k}(q,t_0) - \vec{k}(p,t_0)\big\rangle + \frac{d}{\psi^2\pi}\sin\alpha \int\limits_{\mathbb{S}^1} k^2\,ds \\ &\quad + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds - \frac{dl}{\psi^2 L}\cos\alpha \int\limits_{\mathbb{S}^1} k^2\,ds\,,\end{aligned}$$

where we introduced $\alpha = l\pi/L,\ 0 < \alpha \le \pi/2$. Again we distinguish the two cases from above:

<u>Case 1</u>: $e_1 = e_2$. Using (3.2) we see

$$\begin{aligned}\frac{d}{dt}\Big(\frac{d}{\psi}\Big) &\ge \frac{d}{\psi L}\int\limits_{\mathbb{S}^1} k^2\,ds + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds - \frac{dl}{\psi^2 L}\cos\alpha \int\limits_{\mathbb{S}^1} k^2\,ds \\ &= \frac{d}{\psi L}\Big(1 - \frac{l}{\psi}\cos\alpha\Big)\int\limits_{\mathbb{S}^1} k^2\,ds + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds\ .\end{aligned}$$

Now note that $\frac{l}{\psi}\cos\alpha = \alpha(\tan\alpha)^{-1} < 1$, since $0 < \alpha \le \pi/2$ by assumption. This shows the desired sign on the time derivative.

<u>Case 2</u>: $e_1 \ne e_2$. By (3.3) we have

$$\frac{d}{dt}\Big(\frac{d}{\psi}\Big) \ge -\frac{4\pi^2}{L^2}\frac{d}{\psi} + \frac{d}{\psi L}\Big(1 - \frac{l}{\psi}\cos\alpha\Big)\int\limits_{\mathbb{S}^1} k^2\,ds + \frac{d}{\psi^2}\cos\alpha \int\limits_p^q k^2\,ds\ .$$

Since $\int_{\mathbb{S}^1} k\,ds = 2\pi$ the Hölder inequality gives

$$\int\limits_{\mathbb{S}^1} k^2\,ds \ge \frac{1}{L}\Big(\int\limits_{\mathbb{S}^1} k\,ds\Big)^2 = \frac{4\pi^2}{L},$$

and since as before $\big(1 - \frac{l}{\psi}\cos\alpha\big) > 0$ we obtain

$$\frac{d}{dt}\Big(\frac{d}{\psi}\Big) \geq \frac{d}{\psi^2 l}\cos\alpha\Big(-\frac{4\pi^2 l^2}{L^2} + l\int_p^q k^2\,ds\Big)\,. \tag{3.4}$$

But now notice that

$$l\int_p^q k^2\,ds \geq \Big(\int_p^q k\,ds\Big)^2 = \beta^2$$

where $0 < \beta \leq \pi$ is the angle between e_1 and e_2. Since $e_1 + e_2$ is parallel to ω we have by equation (3.1)

$$\cos\Big(\frac{\beta}{2}\Big) = \langle e_1, \omega\rangle = \langle e_2, \omega\rangle = \frac{d}{\psi}\cos\alpha$$

and since by assumption $(d/\psi)(p, q, t_0) < 1$ we have $\cos(\beta/2) < \cos\alpha$ and thus $\alpha < \beta/2$. Thus

$$l\int_p^q k^2\,ds \geq \beta^2 > 4\alpha^2 = \frac{4\pi^2 l^2}{L^2}\,,$$

which implies the desired inequality. □

Remark 3.1.2: Note that this implies that $\min_{\gamma_t}\frac{d}{\psi} \geq \min_{\gamma_0}\frac{d}{\psi} > 0$ for all $t \in [0, T)$ since we assume that the initial curve is embedded. It is also important to note that this quantity is invariant under scaling.

Exercise 3.1.3: Show that for an embedded, closed self-similarly shrinking curve this implies that $d/\psi \geq 1$. Use this to show that the solution has to be a round, shrinking circle, thus completing the proof of Theorem 2.2.11 as stated there.

3.2 Convergence to a 'round' point

The distance comparison principle from the last section will enable us to show that if (x_0, T) is a singular point of the flow, then any sequence of rescaling

$$\gamma^\lambda_{t'} = \lambda(\gamma_{T+\lambda^{-2}t'} - x_0)\,. \tag{3.5}$$

converges to the homothetically shrinking circle.

We will first show the following weaker convergence result.

Lemma 3.2.1. *Let (x_0, T) be a point reached by the flow. Then for any sequence of rescalings as in (3.5) with $\lambda_i \to \infty$ there exists a subsequence, labeled again the same, such that for almost all $t \in (\infty, 0)$ and for any $\alpha \in (0, 1/2)$*

$$\gamma_t^{\lambda_i} \to \gamma_t^\infty$$

in $C^{1,\alpha}_{loc}$, where γ_t^∞ is either a constant line through the origin or the self-similarly shrinking circle. Furthermore, we have

$$\Theta((\gamma_t^\infty), (0,0), r) = \Theta((\gamma_t), (x_0, T))$$

for all $r > 0$.

Proof. Let

$$f_i(t) := \int\limits_{\gamma_t^{\lambda_i}} \left|\vec{k} - \frac{x^\perp}{2t}\right|^2 \rho_{0,0}(\cdot, t)\, ds.$$

Note that the rescaled monotonicity formuly, see (2.8), implies that $f_i \to 0$ in $L^1_{\mathrm{loc}}((-\infty, 0])$. Thus there exists a subsequence such that f_i converges point-wise a.e. to zero. This implies that for any such t' and $R > 0$

$$\int\limits_{\gamma_t^{\lambda_i} \cap B_R(0)} |k|^2 ds \leq C\,,$$

independent of i. By choosing a further subsequence we can assume that γ_t^i converges in $C^{1,\alpha}_{\mathrm{loc}}$ to a limiting curve. Note first that the distance comparison principle from the last section implies that no higher multiplicities can develop, and the limiting curve is embedded. Note further that each limiting curve is in $W^{2,2}_{\mathrm{loc}}$ and is a weak solution of

$$\vec{\kappa} = \frac{x^\perp}{2t}\,.$$

By elliptic regularity, each such curve is actually smooth, and thus by theorem 2.2.11 the limiting curve is either a straight line through the origin or the centered circle of radius $\sqrt{-2t}$. That the Gaussian density ratios in the limit are equal to the Gaussian density of (γ_t) at (x_0, t_0) follows from Exercise 2.2.2 and the $C^{1,\alpha}_{\text{loc}}$-convergence. $\square$

Note that the Gaussian density of a line through the origin is one, and the Gaussian density of the shrinking circle can be computed to be $\sqrt{2/e} \approx 1.520$. Since the Gaussian density of the limiting flow coincides with the Gaussian density of the initial flow at the point (x_0, T) we only have the following two cases. Either any rescaling subconverges to a line through the origin or all rescalings converge to the shrinking circle - independently of the sequence of rescalings chosen.

Let us first consider the case that $\Theta(x_0, T) = 1$, so any sequence of rescalings has a subsequence which converges for a.e. t to a line through the origin. Note that by using big spheres as barriers we see that the orientation of the limiting line does not depend on t. We can assume w.l.o.g. that the limiting line is the axis $\{x_2 = 0\}$. Again by using spheres as barriers we can actually see that for $\varepsilon > 0$ there exists i_0 such that for $i > i_0$ we have

$$\gamma_t^{\lambda_i} \cap B_{100}(0) \subset \{|x_2| \leq \varepsilon\} \cap B_{100}(0) \qquad \text{for all } t \in [-2, 0).$$

We now fix such an $i > i_0$. We want to show that (x_0, t_0) is a smooth point of the flow, that is, there is a $C > 0$ such that

$$|k|_{\gamma_t^{\lambda_{i_0}}}\Big|_{B_1(0)} \leq C \qquad \text{for all } t \in [-1, 0).$$

(Note that this implies by theorem 2.1.8 that γ_t is smooth in a neighbourhood of (x_0, T) and thus any sequence converges to the same line through the origin).

By the previous lemma, we can assume that there is a $t_0 \in (-3, -2)$ such that $\gamma_{t_0}^{\lambda_i}$ is $C^{1,\alpha}$-close to $\{x_2 = 0\}$ on $B_{100}(0)$. This implies that $\gamma_{t_0}^{\lambda_i}$ can be written as a graph of a function with small gradient over $\{x_2 = 0\}$ on $B_{50}(0)$. Due to the C^1-convergence, the Gaussian densities at t_0

$$\Theta_{t_0}(x, t_0 + r^2) \leq 1 + \varepsilon$$

for all $1 < r < 2$ and $x \in B_{50}(0)$. We can thus apply theorem 2.2.8 to see that the second fundamental form (and all its derivatives) are bounded on $B_{10}(0) \times [t_0 + 1, 0)$.

Note that by the previous reasoning there has to exist a point (x_0, T) such that every rescaling sequence has a subsequence which converges point-wise a.e. to the shrinking circle. Thus we can assume that for every $\varepsilon > 0$ there is a $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ there exits a $t_\lambda \in (-3, -2)$ such that

$$\gamma^\lambda_{t_\lambda} \text{ is } \varepsilon\text{-close to } \sqrt{-2t_\lambda} \cdot \mathbb{S}^1 \text{ in } C^{1,\alpha}\,. \tag{3.6}$$

By using shrinking spheres as barriers, this implies that for ε small enough

$$\gamma^\lambda_t \subset B_{(1+2\varepsilon)\sqrt{-2t}}(0) \setminus B_{(1-2\varepsilon)\sqrt{-2t}}(0) \qquad \text{for all } t \in (-2, -1)\ .$$

This implies that on $(-2, -1)$ the subsequence converges in Hausdorff distance to the shrinking circle. But since every rescaling sequence has such a subsequence, this implies that for every sequence the flow converges on $(-2, -1)$ in Hausdorff distance to the shrinking circle. This already implies that the rescaled curves

$$(T - t)^{-1/2}\big(\gamma_t - x_0\big) \to \mathbb{S}^1_{\sqrt{2}} \tag{3.7}$$

in Hausdorff distance. To prove higher order convergence we can use (3.6) together with the regularity result of White as described above.

Remark 3.2.2: One can show that the convergence in (3.7) is exponential. That is if one considers a new time variable $\tau = -\log(-t)$ then the convergence in (3.7) is actually exponential in every C^k-norm.

4 Evolution of closed, convex hypersurfaces

In this section we will study the evolution of closed, convex hypersurfaces in Euclidean space. We will present a proof Huisken's result below, where we do not follow the original proof, but again make use of the monotonicity formula and and estimate on the inner and outer radius of pinched, convex hypersurfaces by B. Andrews [2].

Theorem 4.0.1 (Huisken [24]). *Any closed, convex hypersurface becomes immediately strictly convex under mean curvature flow and converges in finite time to a 'round' point.*

The proof we present here is considerably shorter than Huisken's original proof. The idea of the proof is similar to the work of B. Andrews [2], but we shorten the proof further by using Huisken's classification of mean convex self-similar solutions.

We have seen that Hamilton's maximum principle for 2-tensors implies that closed, convex hypersurfaces stay convex and become immediately strictly convex. Enclosing the initial hypersurface by a big sphere and applying the avoidance principle we see that any such solution can only exist on a finite time interval. We assume in the following that $n \geq 2$ and that $F : M^n \times [0,T) \to \mathbb{R}^{n+1}, T < \infty$ is a maximal solution.

4.1 Pinching of the principal curvatures

We can assume w.l.o.g. that M_0 is strictly convex. Since M_0 is compact there exists an $\varepsilon > 0$ such that

$$m^i_j := h^i_j - \varepsilon H \delta^i_j \geq 0 \ ,$$

where the inequality is understood in the sense of $m^i{}_j$ being positive semi-definite (or equivalently all eigenvalues being non-negative).

Lemma 4.1.1. *If initially $h^i_j - \varepsilon H \delta^i_j \geq 0$, then this is preserved under the flow.*

Proof. Using the evolution equation for h^i_j and H we see that

$$\frac{\partial}{\partial t} m^i_j = \Delta m^i_j + |A|^2 m^i_j \ .$$

Thus by the maximum principle $m^i_j \geq 0$ is preserved. □

This implies that at every point $p \in M$ it holds

$$\lambda_1(p,t) \geq \varepsilon H(p,t) \geq \varepsilon \lambda_n(p,t) \ ,$$

i.e. the principal curvatures are pinched. Recall the Gauss map $\nu : M \to \mathbb{S}^n$ and it's derivative, the Weingarten map

$$W = \bar{\nabla}\nu : T_pM \to T_pM \ ,$$

where we identified $T_{\nu(p)}\mathbb{S}^n$ with T_pM. Thus for strictly convex hypersurfaces the Gauss map is a local diffeomorphism. Even more it is a global diffeomorphism, and we can parametrise the hypersurface by its Gauss map. All information about the hypersurface is contained in the *support function* defined as

$$s(z) = \langle z, F(\nu^{-1}(z)) \rangle \tag{4.1}$$

for all $z \in \mathbb{S}^n$. Note that in the standard parametrisation via F the support function is just $s(p) = \langle \nu, x \rangle = \langle \nu(p), F(p) \rangle$. If the support function is known, the hypersurface is given as the boundary of the convex region $\cap_{z\in\mathbb{S}^n}\{y \in \mathbb{R}^{n+1} \mid \langle y, z\rangle \leq s(z)\}$.

Exercise 4.1.2: Define the map $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ by

$$f(z) = s(z)z + \tilde{\nabla}s(z)\,,$$

where $\tilde{\nabla}$ is the standard covariant derivative on $\mathbb{S}^n$.

(i) Show that $F(p) = f(\nu(p))$ for all $p \in M^n$ if s comes from a strictly convex immersion $F : M^n \to \mathbb{R}^{n+1}$.

(ii) Show that for $U, V \in T_\nu\mathbb{S}^n$ it holds

$$A(W^{-1}(U), W^{-1}(V)) = (\tilde{\nabla}^2 s + s\,\tilde{g})(U, V)\,,$$

where $\tilde{g}$ is the standard metric on $\mathbb{S}^n$ and we consider the Weingarten map as a map $W : T_pM \to T_{\nu(p)}\mathbb{S}^n$.

The support function provides some useful descriptions of the general shape of a convex hypersurface. For example the *width function* is defined on $\mathbb{S}^n$ by $w(z) = s(z) + s(-z)$. This gives the separation of the tangent planes at the points $f(z)$ and $f(-z)$, since these two planes are parallel. We denote the maximum and minimum widths by w_+ and w_-, respectively.

Lemma 4.1.3 (Andrews, [2], Lemma 5.1)**.** *Let $F : M^n \to \mathbb{R}^{n+1}$ be a strictly convex embedding of a compact manifold M^n such that there exists $C_1 > 0$ such that at every point $p \in M^n$*

$$\lambda_n(p) \leq C_1\lambda_1(p)\ . \tag{4.2}$$

Then

$$w_+ \leq C_1 w_-\ .$$

Proof. First note that the eigenvalues of the symmetric (0,2)-tensor

$$\tilde{A} = \tilde{\nabla}^2 s + s\,\tilde{g}$$

also satisfy a pinching condition with respect to $\tilde{g}$ with the same constant C_1. Choose $z_+, z_- \in \mathbb{S}^n$ such that $w_+ = s(z_+) + s(-z_+)$ and $w_- = s(z_-) + s(-z_-)$. Let Σ be any totally geodesic 2-sphere in $\mathbb{S}^n$ which contains both z_+ and z_-.

Define two sets of standard spherical coordinates $(\phi_\pm, \theta_\pm)$ on $\Sigma : \phi_\pm(z) = \cos^{-1}\langle z, z_\pm\rangle$, and $\theta_\pm$ is the angle around a great circle perpendicular to $z_\pm$. the following calculation gives expressions for the maximum and minimum width of $F(M)$:

$$\begin{aligned}
\int_\Sigma \tilde{A}\left(\frac{\partial}{\partial\phi_\pm}, \frac{\partial}{\partial\phi_\pm}\right) d\mu_\Sigma &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} (\tilde{\nabla}_{\phi_\pm}\tilde{\nabla}_{\phi_\pm} s + s)\cos\phi_\pm \, d\phi_\pm \, d\theta_\pm \\
&= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \left(\frac{\partial^2}{\partial\phi_\pm^2} s + s\right)\cos\phi_\pm \, d\phi_\pm \, d\theta_\pm \\
&= 2\pi\left(s(z_\pm) + s(-z_\pm)\right) \ ,
\end{aligned}$$

where we used that the curves $\theta_\pm = \text{const}$ are geodesics on $\mathbb{S}^n$ and thus the Hessian in direction $(\frac{\partial}{\partial\phi_\pm}, \frac{\partial}{\partial\phi_\pm})$ is equal to the second partial derivatives, and we integrated by parts twice. Note that $\frac{\partial}{\partial\phi_+}$ and $\frac{\partial}{\partial\phi_-}$ have unit length almost everywhere with respect to $\tilde{g}$, so $\tilde{A}\left(\frac{\partial}{\partial\phi_+}, \frac{\partial}{\partial\phi_+}\right) \leq C_1\tilde{A}\left(\frac{\partial}{\partial\phi_-}, \frac{\partial}{\partial\phi_-}\right)$ almost everywhere. ☐

We define the inner radius ρ_- and the outer radius ρ_+ by

$$\begin{aligned}
\rho_+(t) &= \inf\{r : B_r(y) \text{ encloses } M_t \text{ for some } y \in \mathbb{R}^{n+1}\} \\
\rho_-(t) &= \sup\{r : B_r(y) \text{ is enclosed by } M_t \text{ for some } y \in \mathbb{R}^{n+1}\}\,.
\end{aligned}$$

The following lemma relates the maximum and minimum width to the outer and inner radius.

Lemma 4.1.4 (Andrews, [2], Lemma 5.4)**.** *For any compact, convex hypersurface, the following estimates hold:*

$$\rho_+ \leq \frac{w_+}{\sqrt{2}} \quad \textit{and} \quad \rho_- \geq \frac{w_-}{n+2}\ .$$

Consequently, if the pinching estimate (4.2) holds, we have $\rho_+ \leq C_2\rho_-$ for some constant C_2.

Proof. Let Σ be a sphere of smallest radius which encloses $F(M)$, and assume it has centre at the origin. Let $S = S \cap F(M)$, and assume that z_0 and z_1 are two points in S which maximise the distance $|z_0 - z_1|$. Clearly the angle between z_0 and z_1 is obtuse, since otherwise Σ could be moved to strictly contain $F(M)$, contradicting the assumption that Σ has smallest possible radius. Then the distance from z_0 to z_1 is a lower bound for the maximum width w_+, and is at least $\sqrt{2}$ times the radius of Σ, or $\sqrt{2}\rho_+$.

Now let Σ be a sphere of largest radius enclosed by $F(M)$, and choose the origin at the centre of Σ. Let $S = \Sigma \cap F(M)$. One can show that there is a nonempty set of points $P \subset S$ such that $P \setminus z$ is linearly independent for any $z \in P$, and such that there is a positive linear combination of the elements of P with value zero - if this were not the case, then the convex hull of S could not contain the origin, and so Σ could be moved slightly to become properly contained by $F(M)$. Let E be the smallest affine subspace of $\mathbb{R}^{n+1}$ which contains the set P. Note that E has dimension $k-1$, where P has k elements. Let $\bar{S}$ be the simplex $\{y \in E \,|\, \langle y, z\rangle \leq s(z) \text{ for all } z \in P\}$. By convexity, $\bar{S}$ contains the projection of $F(M)$ onto E. Hence the minimum width of $F(M)$ is less than the minimum width of $\bar{S}$, which is the shortest altitude of $\bar{S}$. This is bounded by the altitude of a regular simplex inscribed by Σ in E, or $k\rho_-$. Since E has dimension at most $n+1$, the result follows. □

4.2 Convergence to a 'round' point

We will first show that the solution exists as long as it bounds a ball of radius $\delta > 0$. Note that since all the surfaces are convex we have $|A|^2 \leq H^2$.

Proposition 4.2.1. *Assume that M_t encloses $B_\delta(0)$ for $t \in [0, t']$. Then*

$$H(t) \leq 2\rho_+(t) \max\left\{\frac{8n}{\delta^2}, \frac{2\sup_{M_0} H}{\delta}\right\},.$$

Proof. Since all M_t enclose $B_\delta(0)$ for $t \in [0, t']$ and are convex, we have

$$\langle x, \nu\rangle \geq \delta\ .$$

The evolution equation of $\langle x, \nu\rangle$ is given by

$$\frac{\partial}{\partial t}\langle x, \nu\rangle = \Delta\langle x, \nu\rangle + |A|^2\langle x, \nu\rangle - 2H \ .$$

Let $\beta = \delta/2$, then we have $\langle x, \nu\rangle - \beta \geq \beta$. We define the function

$$v = \frac{H}{\langle x, \nu\rangle - \beta}$$

which satisfies, using that $|A|^2 \geq \frac{1}{n}H^2$

$$\begin{aligned}\frac{\partial}{\partial t}v &= \Delta v + \frac{2}{\langle x, \nu\rangle - \beta}\big\langle\nabla\langle x, \nu\rangle, \nabla v\big\rangle + 2v^2 - \beta\frac{|A|^2}{H}v^2 \\ &\leq \Delta v + \frac{2}{\langle x, \nu\rangle - \beta}\big\langle\nabla\langle x, \nu\rangle, \nabla v\big\rangle + \big(2 - \frac{\beta}{n}H\big)v^2\end{aligned}$$

Let us assume that v attains a new maximum which is greater than C at a point (p, t). Then we have at this point $H > \beta C$ and we get a contradiction if

$$C \geq \frac{2n}{\beta^2} \ .$$

Thus we obtain

$$H \leq \max\Big\{\frac{2n}{\beta^2}, \frac{\sup_{M_0} H}{\beta}\Big\}\big(\langle x, \nu\rangle - \beta\big) \leq 2\rho_+(t)\max\Big\{\frac{2n}{\beta^2}, \frac{\sup_{M_0} H}{\beta}\Big\}$$

□

By the previous proposition, together with Lemma 4.1.4, we see that the solution exists until $\rho_- \to 0$. Furthermore the solution contracts for $t \to T$ to a point x_0.

Lemma 4.2.2 (Andrews, [2]). *We have with C_2 as in Lemma 4.1.4:*

$$C_2^{-1}\sqrt{2n(T - t)} \leq \rho_-(t) \, .$$

Proof. Let y be such that $\mathbb{S}_{\rho_+(t)}(y)$ encloses M_t. By the avoidance principle $M_{t'}$ remains enclosed by $\mathbb{S}_{\rho(t')}(y)$ for all t' in the range (t, T), where $\rho(t') =$

$\sqrt{\rho_+^2(t) - 2n(t'-t)}$. Thus

$$\rho_+^2(t') \le \rho_+^2(t) - 2n(t'-t) \ .$$

Since the solution exists until $t' = T$ we have

$$\rho_+^2(t) \ge 2n(T-t) \ \Rightarrow \ \rho_-^2(t) \ge C_2^{-2}\, 2n(T-t) \ .$$

$\square$

Applying this to the proposition before on $[0,t)$, with

$$\delta = \rho_-(t) \ge C_2^{-1}\sqrt{2n(T-t)},$$

we see that we have for t sufficiently close to T that

$$|A|(t) \le H(t) \le 16n\, \frac{\rho_+(t)}{(\rho_-(t))^2} \le \frac{C}{\rho_-(t)} \le \frac{C}{\sqrt{T-t'}} \ ,$$

and thus the singularity is of type I. By Exercise 2.2.6 any sequence of rescalings

$$M_{t'}^{\lambda_i} = \lambda_i(M_{T+\lambda_i^{-2}t'} - x_0)$$

for $\lambda_i \to \infty$ converges, up to a subsequence, smoothly on any compact sub-interval of $(-\infty, 0)$ to a convex, selfsimilar solution. Note that the limiting solution still satisfies $\rho_+ \le C_1\rho_-$ and thus it can only be the shrinking sphere by Theorem 2.2.10 . Since this is true for any sequence of rescalings, we obtain that for every fixed $t' < 0$ we have

$$\lambda(M_{T+\lambda^{-2}t'} - x_0) \to \sqrt{-t'} \cdot \mathbb{S}^n_{\sqrt{2n}}$$

smoothly as $\lambda \to \infty$. Thus choosing

$$\lambda(t) = \left(\frac{-t'}{T-t}\right)^{\frac{1}{2}}$$

we see that

$$\frac{1}{\sqrt{T-t}}\big(M_t - x_0\big) \to \mathbb{S}^n_{\sqrt{2n}}$$

in C^∞.

Remark 4.2.3: As before for curves one can also show that the convergence is exponential.

5 Mean convex mean curvature flow

In this chapter we aim to study the singularity behaviour for mean convex mean curvature flow. By Huisken's classification of mean convex self-shrinkers, Theorem 2.2.10, we expect that if the curvature is large, then the flow is close to $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $k \in \{1, \ldots, n-1\}$. In the following we will prove estimates of Huisken-Sinestrari which give quantitative estimates confirming this expectation.

We will first collect and recall some basic properties of mean convex mean curvature flow. Recall the evolution equations for H and $|A|^2$:

$$\frac{\partial}{\partial t} H = \Delta H + H|A|^2$$

and

$$\frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 \ .$$

Proposition 5.0.1. *Let $(M_t)_{0 \le t < T}$ be a family of closed hypersurfaces moving my mean curvature flow.*

(i) If $H \geq 0$ on M_0, then $H > 0$ on M_t for $t > 0$.

(ii) If $|A|^2 \leq CH^2$ on M_0 then $|A|^2 \leq CH^2$ on M_t for $t > 0$

Proof. Part (i) follows from the evolution equation and the strong maximum principle, see Theorem 2.1.2. For (ii), we compute the evolution equation for

$f = |A|^2/H^2$:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{H^2}\frac{\partial}{\partial t}|A|^2 - 2\frac{|A|^2}{H^3}\frac{\partial}{\partial t}H \\ &= \frac{1}{H^2}\Delta|A|^2 - 2\frac{|A|^2}{H^3}\Delta H - 2\frac{1}{H^2}|\nabla A|^2 \\ &= \Delta f + \frac{2}{H}\langle\nabla H, \nabla f\rangle - \frac{2}{H^4}|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2 .\end{aligned}$$

This follows from

$$\begin{aligned}\Delta\frac{|A|^2}{H^2} &= \frac{1}{H^2}\Delta|A|^2 + |A|^2\Delta\frac{1}{H^2} + 2\left\langle\nabla|A|^2, \nabla\frac{1}{H^2}\right\rangle \\ &= \frac{1}{H^2}\Delta|A|^2 - 2\frac{|A|^2}{H^3}\Delta H + 6\frac{|A|^2}{H^4}|\nabla H|^2 - 4\frac{1}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle \\ &= \frac{1}{H^2}\Delta|A|^2 - 2\frac{|A|^2}{H^3}\Delta H + 6\frac{|A|^2}{H^4}|\nabla H|^2 - 2\frac{1}{H^3}\left\langle\nabla|A|^2, \nabla H\right\rangle \\ &\quad - \frac{2}{H}\langle\nabla f, \nabla H\rangle - 4\frac{|A|^2}{H^4}|\nabla H|^2\end{aligned}$$

and the identity

$$|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2 = H^2|\nabla A|^2 - H\langle\nabla|A|^2, \nabla H\rangle + |A|^2|\nabla H|^2 .$$

The statement then follows from the maximum principle. □

Corollary 5.0.2. *Let $(M_t)_{0\leq t<T}$ be a family of closed hypersurfaces moving my mean curvature flow. If $H > 0$ on M_0, then there exists an $\varepsilon_0 > 0$ such that*

$$\varepsilon_0|A|^2 \leq H^2 \leq n|A|^2$$

on M_t for all $0 \leq t < T$.

Proof. By compactness of M_0, if $H > 0$ everywhere then we also have $H^2 \geq \varepsilon_0|A|^2$ everywhere for some ε_0. Thus by the previous proposition this is preserved under the flow. The estimate $H^2 \leq n|A|^2$ follows since by Cauchy-Schwarz

$$H = \sum_{i=1}^{n}\lambda_i \leq n^{1/2}|A|^{1/2} .$$

□

We will present some further invariant curvature condition under mean curvature flow. For that we need a refined version of Hamilton's maximum principle.

Theorem 5.0.3. *Let M be closed and m^i_j be a symmetric bilinear form, which solves*

$$\frac{\partial m^i_j}{\partial t} = \Delta m^i_j + \phi^i_j(m^i_j)\,,$$

where ϕ^i_j is a symmetric bilinear form, depending on m^i_j. Assume that the convex $O(n)$-invariant cone C in the space of symmetric bilinear forms is preserved by the ODE

$$\frac{\partial m^i_j}{\partial t} = \phi^i_j(m^i_j)\,,$$

then C is also preserved by the full PDE.

For a proof see again [18, Lemma 8.2].

We will say that an immersed hypersurface M is $k-$*convex* for some $1 \leq k \leq n$, provided

$$\lambda_1 + \cdots + \lambda_k \geq 0$$

at every point in M. In particular 1-convexity coincides with convexity, while n-convexity is equivalent to $H \geq 0$.

Proposition 5.0.4. *If M_0 satisfies $\lambda_1 + \cdot + \lambda_k \geq \alpha H$ for some $\alpha \geq 0$ and $1 \leq k \leq n$, then this is preserved under mean curvature flow. In particular if M_0 is k-convex then so is M_t.*

Proof. The result follows from Hamilton's maximum principle for tensors, provided we show that the inequality $\lambda_1 + \cdot + \lambda_k \geq \alpha H$ describes a convex cone in the set of all matrices, and that this cone is invariant under the system of ODEs

$$\frac{\partial}{\partial t} h^i{}_j = |A|^2 h^i{}_j\ ,$$

which is obtained from the evolution equation of the Weingarten operator $h^i{}_j$ by dropping the diffusion term. If we denote bu $W(v_1, v_2)$ the Weingarten operator applied to two tangent vectors v_1, v_2 at any point, we have

$$\lambda_1 + \cdots + \lambda_k = \min\{W(e_1, e_1) + \cdots W(e_k, e_k)\,|\,\langle e_i, e_j\rangle = \delta_{ij} \text{ for all } 1 \leq i \leq j \leq k\}$$

This shows that $\lambda_1 + \cdots + \lambda_k$ is a concave function of the Weingarten operator, being the infimum of a family of linear maps. Therefore the inequality $\lambda_1 + \cdots + \lambda_k \geq \alpha H$ describes a convex cone of matrices. In addition, the vector field $|A|^2 h^i{}_j$ is pointwise a multiple of $h^i{}_j$, which corresponds to scaling, and thus the ODE $\frac{\partial}{\partial t} h^i{}_j = |A|^2 h^i{}_j$ leaves any cone invariant. □

5.0.1 Convexity and cylindrical estimates

We have seen in the last paragraph that uniform two-convexity is preserved under mean curvature flow. Thus we will in the following assume (without mentioning it always) that we assume that $H > 0$ and that there exists $\alpha > 0$ such that

$$\lambda_1 + \lambda_2 \geq \alpha H \ .$$

Exercise 5.0.5: Show that this assumption implies that $|A|^2 \leq nH^2$ and $\lambda_i \geq \frac{\alpha}{2} H$ for $i = 2, \ldots, n$.

We will in the following present an alternative proof of Huisekn-Sinestrari's convexity and cylindrical estimates for two-convex mean curvature flow which follows a recent approach of Huy Nguyen. We are grateful to Huy for pointing out this alternative approach, which shortens the original estimates of Huisken-Sinestrari significantly. The original proof of Huisken-Sinestrari first proves the asymptotic convexity [28, 29] using a complex procedure through induction on elementary symmetric polynomials utilising the Michael-Simon's inequality and Stampacchia iteration. The asymptotic convexity is then used to bound the curvature term in the Simon's identity from below with a positive term to first order. An alternative procedure is given by White [38, 39] (see also Haslhofer-Kleiner [22, 23] using weak versions of the mean curvature flow - the level set flow and Brakke solutions). We will prove the cylindrical estimate directly from two convexity. The convexity result can then be shown to be a consequence of the cylindrical result.

We consider again the quotient $|A|^2/H^2$ as in the proof of Theorem 5.0.1. Observe that in a cylinder $\mathbb{R}\times\mathbb{S}^{n-1}$ we have $|A|^2/H^2 \equiv 1/(n-1)$. A kind of converse implication also holds, namely: if at one point we have $|A|^2/H^2 = 1/(n-1)$ and in addition $\lambda_1 = 0$, then necessarily $\lambda_2 = \cdots = \lambda_n$. In fact we have the identity

$$|A|^2 - \frac{1}{n-1}H^2 = \frac{1}{n-1}\left(\sum_{1<i<j\le n}(\lambda_i-\lambda_j)^2 + \lambda_1(n\lambda_1 - 2H)\right) , \tag{5.1}$$

which follows directly from (1.1).

Poincaré type inequality

We recall Simon's identity (1.8)

$$\nabla_k\nabla_l h_{ij} - \nabla_i\nabla_j h_{kl} = h_{kl}h_i^{\ p}h_{pj} - h_{il}h_{kp}h^p_{\ j} + h_{kj}h_i^{\ p}h_{pl} - h_{ij}h_k^{\ p}h_{pl}$$

We symmetrise in k,l and i,j to get

$$\begin{aligned}\nabla_k\nabla_l h_{ij} + \nabla_l\nabla_k h_{ij} - \nabla_i\nabla_j h_{kl} - \nabla_j\nabla_i h_{kl} = \\ &= h_{kl}h_i^{\ p}h_{pj} - h_{il}h_{kp}h^p_{\ j} + h_{kj}h_i^{\ p}h_{pl} - h_{ij}h_k^{\ p}h_{pl} \\ &\quad + h_{lk}h_j^{\ p}h_{pi} - h_{jk}h_{lp}h^p_{\ i} + h_{li}h_j^{\ p}h_{pk} - h_{ji}h_l^{\ p}h_{pk} \\ &= 2h_{kl}h_i^{\ p}h_{pj} - 2h_{ij}h_k^{\ p}h_{pl}\end{aligned}$$

We let $C_{ijkl} = h_{kl}h_i^{\ p}h_{pj} - h_{ij}h_k^{\ p}h_{pl}$ and trace both sides with respect to C_{ijkl}. On the right hand side we get $2|C|^2$. We compute this term explicitely

$$|C|^2 = (h_{kl}h^2_{ij} - h_{ij}h^2_{kl})(h^{kl}(h^{ij})^2 - h^{ij}(h^{kl})^2) = 2|A|^2\mathrm{tr}(A^4) - 2\,\mathrm{tr}(A^3)^2 .$$

Diagonalising the second fundamental form we see that

$$\begin{aligned}\sum_{i,j=1}^n(\lambda_i-\lambda_j)^2\lambda_i^2\lambda_j^2 &= \sum_{i,j=1}^n(\lambda_i^2+\lambda_j^2-2\lambda_i\lambda_j)\lambda_i^2\lambda_j^2 = \sum_{i,j=1}^n(\lambda_i^4\lambda_j^2+\lambda_j^4\lambda_i^2-2\lambda_i^3\lambda_j^3) \\ &= 2|A|^2\mathrm{tr}(A^4) - 2\,\mathrm{tr}(A^3)^2 = |C|^2 .\end{aligned}$$

Note that C is symmetric in i,j and k,l. This implies

$$
\begin{aligned}
&2(\nabla_k\nabla_l h_{ij} - \nabla_i\nabla_j h_{kl})C^{ijkl} = \\
&\qquad = (\nabla_k\nabla_l h_{ij} + \nabla_l\nabla_k h_{ji} - \nabla_i\nabla_j h_{kl} - \nabla_j\nabla_i h_{lk})C^{ijkl} \\
&\qquad = 2|C|^2 = 2\sum_{i,j=1}^{n}(\lambda_i - \lambda_j)^2\lambda_i^2\lambda_j^2\,.
\end{aligned}
\tag{5.2}
$$

Now we wish to show that when a point is not cylindrical, i.e. $|A|^2 - \frac{1}{n-1}H^2 \neq 0$ and $\lambda_1 + \lambda_2 > 0$ then $|C|^2 > 0$. Hence we need only to analyse $|C|^2 = 0$, that is when

$$\sum_{i,j=1}^{n}(\lambda_i - \lambda_j)^2\lambda_i^2\lambda_j^2 = 0\,.$$

This implies that for each pair $i \neq j$ we have either $\lambda_i = \lambda_j$ or $\lambda_i = 0$ or $\lambda_j = 0$. Note that $\lambda_1 + \lambda_2 > 0$ implies that $\lambda_2 > 0$ and thus $\lambda_j > 0$ for $j \geq 2$. But this already implies that either $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \kappa > 0$ or $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \kappa > 0$ and $\lambda_1 = 0$.

We will need the following Poincaré-type inequality.

Lemma 5.0.6. *Let $n \geq 3, \alpha \in (0,1)$ and $\eta \in (0,(n-1)^{-1/2} - n^{-1/2})$. Then there exists $\gamma = \gamma(n,\alpha,\eta)$ with the following property: Let $F : M^n \to \mathbb{R}^{n+1}$ be a mean convex, uniformly two convex hypersurface, i.e. $\lambda_1 + \lambda_2 \geq \alpha H$. Let*

$$f_\eta := |A| - \frac{1}{\sqrt{n-1}}H - \eta H$$

and consider the set

$$U_{\eta,M} = \{x \in M \mid f_\eta \geq 0\}\,.$$

Assume $u \in W^{2,2}(M)$ sucht that $\operatorname{spt} u \subset U_{\eta,M}$. Then for any $r \geq 1$ it holds

$$\gamma\int u^2|A|^2\,d\mu \leq r^{-1}\int |\nabla u|^2\,d\mu + (1+r)\int u^2\frac{|\nabla A|^2}{H^2}\,d\mu\,.$$

Proof. We claim that

$$\gamma(n,\alpha,\eta)|A|^2H^4 \leq |C|^2 \qquad \text{on} \quad U_\eta\,. \tag{5.3}$$

This follows by a rescaling an compactness result. Indeed, if this is not true, then there exists a sequence of points $\lambda^l = (\lambda^l_1, \cdots, \lambda^l_n) \in \mathbb{R}^n$ satisfying $\mathrm{tr}(\lambda^l) > 0$ as well as

$$f_\eta(\lambda^l) := |\lambda^l| - \frac{1}{\sqrt{n-1}}\mathrm{tr}(\lambda^l) - \eta\,\mathrm{tr}(\lambda^l) \geq 0$$

and

$$\lambda^l_1 + \lambda^l_2 \geq \alpha\,\mathrm{tr}(\lambda^l)\ ,$$

but

$$\frac{|C(\lambda^l)|^2}{W(\lambda^l)} \to 0 \tag{5.4}$$

as $l \to \infty$, where $W(\lambda^l) = |\lambda^l|^2\mathrm{tr}(\lambda^l)^4$ and

$$|C(\lambda)|^2 := \sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2\lambda_i^2\lambda_j^2\ .$$

Note that by Exercise 5.0.5 and the inequality $|\lambda|^2 \geq \frac{1}{n}\,\mathrm{tr}(\lambda)^2$ we have

$$|\lambda^l|^6\frac{1}{n^2} \leq W(\lambda) \leq n\,\mathrm{tr}(\lambda)^6\ .$$

We take $r_l := W(\lambda_l)^{-1/6}$ and define $\hat\lambda^l = r_l\lambda^l$. Note that $W(\hat\lambda) = 1$ and thus

$$|\hat\lambda^l|^2 \leq n^{2/3}$$

as well as

$$\mathrm{tr}(\hat\lambda^l) \geq \frac{1}{n^{1/6}}\ .$$

We can thus assume, that up to subsequence, $\hat\lambda^l \to \hat\lambda \in \mathbb{R}^n$. Note that $\hat\lambda$ still satsifies

$$|\hat\lambda| - \frac{1}{\sqrt{n-1}}\mathrm{tr}(\hat\lambda) - \eta\,\mathrm{tr}(\hat\lambda) \geq 0 \tag{5.5}$$

as well as

$$\hat\lambda_1 + \hat\lambda_2 \geq \alpha\,\mathrm{tr}(\hat\lambda) > 0\ ,$$

but (5.4) implies

$$|C(\hat\lambda)|^2 = 0.$$

Thus the discussion earlier implies that either

$$\hat\lambda_1 = \hat\lambda_2 = \cdots = \hat\lambda_n = \kappa > 0$$

or

$$\hat\lambda_1 = 0 \text{ and } \hat\lambda_2 = \cdots = \hat\lambda_n = \kappa > 0\ .$$

Using that $\mathrm{tr}(\hat\lambda) > 0$, we see that both cases contradict (5.5), which proves (5.3). Using (5.2) and (5.3), we can estimate

$$\begin{aligned}
\gamma \int u^2|A|^2\, d\mu &\leq \int u^2 H^{-4}|C|^2\, d\mu = \int u^2 H^{-4} C^{ijkl}(\nabla_k\nabla_l h_{ij} - \nabla_i\nabla_j h_{kl})\, d\mu \\
&= \int u^2 \left(2H^{-4}C^{ijkl}\frac{\nabla_i u}{u} - 4C^{ijkl}\frac{\nabla_i H}{H^5} + H^{-4}\nabla_i C^{ikjl}\right)\nabla_j h_{kl}\, d\mu \\
&\quad - \int u^2 \left(2H^{-4}C^{ijkl}\frac{\nabla_k u}{u} - 4C^{ijkl}\frac{\nabla_k H}{H^5} + H^{-4}\nabla_k C^{ikjl}\right)\nabla_l h_{ij}\, d\mu \\
&\leq C\int u^2\left(\frac{|\nabla u|}{u} + \frac{|\nabla A|}{H}\right)\frac{|\nabla A|}{H}\, d\mu
\end{aligned}$$

where C denotes a constant which only depends on n. The claim then follows from Young's inequality. □

Cylindrical estimates

We recall the evolution equation for the $|A|^2$

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\ .$$

Note that since $|A|^2 \geq \frac{1}{n}H^2 > 0$ the function $|A|$ is a smooth function along a uniformly two-convex mean curvature flow and we can compute its evolution equation (exercise)

$$
\begin{aligned}
\frac{\partial}{\partial t}|A| &= \Delta|A| - \frac{1}{2|A|^3}|h_{ij}\nabla_k h_{lm} - h_{lm}\nabla_i h_{jk}|^2 + |A|^3 \\
&= \Delta|A| - \frac{1}{2|A|^3}|A\otimes\nabla A - \nabla A\otimes A|^2 + |A|^3
\end{aligned} \tag{5.6}
$$

We want to make use of this good gradient term.

Lemma 5.0.7 (See Lemma 2.1 in [31] and Lemma 2.3 in [24]). *Let $F : M^n \to \mathbb{R}^{n+1}$ be a strictly two convex immersion, i.e. $\lambda_1 + \lambda_2 \geq \alpha H > 0$ for some $\alpha \in (0, 1)$. Then there is a constant $\gamma = \gamma(\alpha, n) > 0$ such that*

$$
|A\otimes\nabla A - \nabla A\otimes A|^2 \geq \gamma|A|^2|\nabla A|^2 \ .
$$

Proof. Pick $x \in M$ such that $|\nabla A| \neq 0$. Multiplying the desired inequality by $|A|^{-2}|\nabla A|^{-2}$ we can assume that $|A| = 1$ and $|\nabla A| = 1$ at x. Note that the set

$$
\{(W, T) \in \mathrm{Sym}_2 \times \mathrm{Sym}_3 \,|\, \lambda_1(W) + \lambda_2(W) \geq \alpha\,\mathrm{tr}(W) \geq 0, |W| = |T| = 1\}\,,
$$

where Sym_k is the set of totally symmetric $(0, k)$-tensors, is compact. Furthermore, the assumptions, as in Exercise 5.0.5 imply $\mathrm{tr}(W) \geq n^{-1/2}|W| = n^{-1/2}$. Thus it suffices to show that

$$
|A\otimes\nabla A - \nabla A\otimes A|^2 > 0\,.
$$

Therefore, assume that we have $A\otimes\nabla A = \nabla A\otimes A$. We choose a diagonalising frame for A and apply the Codazzi equations to get

$$
\lambda_i\delta_{ij}\nabla_k h_{lm} = \lambda_l\delta_{lm}\nabla_k h_{ij}
$$

for each i, j, k, l, m. Now by two-convexity, we have $\lambda_n > 0$. Fix k, l, m such that $\nabla_k h_{lm} \neq 0$. Then we have

$$
\lambda_n\nabla_k h_{lm} = \lambda_l\delta_{lm}\nabla_k h_{nn} \ ,
$$

which implies that $l = m$. Again by the Codazzi equations we see that also

$k = l = m$. Thus $\nabla_k h_{lm}$ is only non-zero if $k = l = m$. This yields

$$\lambda_n \nabla_k h_{kk} = \lambda_k \nabla_k h_{nn} \ ,$$

and thus $k = n$. That is $\lambda_n \nabla_k h_{lm} \neq 0$ if and only if $n = k = l = m$. On the other hand for any $i \neq n$ we get

$$\lambda_i \nabla_n h_{nn} = \lambda_n \nabla_n h_{ii} = 0 \, .$$

Therefore $\lambda_i = 0$ unless $i = n$, but two convexity implies that $\lambda_2 > 0$, so this cannot occur. $\square$

To derive the cylindrical estimate, we consider for $\eta \geq 0$ and $\sigma \in [0,1]$ the following function

$$G_{\sigma,\eta} = \frac{|A| - \left(\frac{1}{\sqrt{n-1}} + \eta\right) H}{H^{1-\sigma}} .$$

We aim to show that for every $\eta > 0$ there is a $\sigma > 0$ such that this function is bounded from above by a constant $C(\sigma,\eta)$. Note that this implies that when the mean curvature is large, the surface is nearly cylindrical. The evolution equation for $G_{\sigma,\eta}$ is given by (exercise)

$$\begin{aligned}
\frac{\partial}{\partial t} G_{\sigma,\eta} = \Delta G_{\sigma,\eta} &+ \frac{2(1-\sigma)}{H}\langle \nabla G_{\sigma,\eta}, \nabla H\rangle - \frac{1}{2H^{1-\sigma}|A|^3}|A \otimes \nabla A - \nabla A \otimes A|^2 \\
&- \frac{\sigma(1-\sigma) G_{\sigma,\eta}}{H^2}|\nabla H|^2 + \sigma |A|^2 G_{\sigma,\eta} \\
&\leq \Delta G_{\sigma,\eta} - \frac{\gamma_1 G_{\sigma,\eta}}{H^2}|\nabla A|^2 + 2|\nabla G_{\sigma,\eta}|\frac{|\nabla H|}{H} + \sigma |A|^2 G_{\sigma,\eta} \ ,
\end{aligned}$$

where we used to previous lemma to estimate the gradient term. Note that the maximum principle nearly gives the desired result up to lowest order term. The idea is now to use integral estimates and the good gradient terms to control the lowest order term.

We let $G_{\sigma,\eta,+} = \max\{G_{\sigma,\eta}, 0\}$ and compute the following evolution equation

$$\frac{d}{dt}\int G^p_{\sigma,\eta,+}\, d\mu = p \int G^{p-1}_{\sigma,\eta,+} \frac{\partial}{\partial t} G_{\sigma,\eta}\, d\mu - \int G^p_{\sigma,\eta,+} H^2\, d\mu \, .$$

We discard the second term and get

$$\frac{d}{dt}\int G^p_{\sigma,\eta,+}\,d\mu \le -p(p-1)\int G^{p-2}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|^2\,d\mu - \gamma_1 p\int G^p_{\sigma,\eta,+}\frac{|\nabla A|^2}{H^2}\,d\mu$$
$$+ 2p\int G^{p-1}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|\frac{|\nabla H|}{H}\,d\mu$$
$$+ \sigma p\int G^p_{\sigma,\eta,+}|A|^2\,d\mu\ .$$

We use Young's inequality to estimate the term

$$2p\int G^{p-1}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|\frac{|\nabla H|}{H}\,d\mu \le p^{3/2}\int G^{p-2}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|^2\,d\mu$$
$$+ Cp^{1/2}\int G^p_{\sigma,\eta,+}\frac{|\nabla A|^2}{H^2}\,d\mu$$

to get

$$\begin{aligned}\frac{d}{dt}\int G^p_{\sigma,\eta,+}\,d\mu \le{}& -(p^2-p^{3/2}-p)\int G^{p-2}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|^2\,d\mu\\ &-(\gamma_1 p - Cp^{1/2})\int G^p_{\sigma,\eta,+}\frac{|\nabla A|^2}{H^2}\,d\mu\\ &+\sigma p\int G^p_{\sigma,\eta,+}|A|^2\,d\mu\ .\end{aligned} \tag{5.7}$$

We use the Poincaré inequality, Lemma 5.0.6, with $u^2 = G^p_{\sigma,\eta,+}$, $r = p^{1/2}$ so that

$$|\nabla u|^2 = \frac{p^2}{4}G^{p-2}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|^2$$

to get

$$\gamma_2\int G^p_{\sigma,\eta,+}|A|^2\,d\mu \le \frac{p^{3/2}}{4}\int G^{p-2}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|^2\,d\mu + (p^{1/2}+1)\int G^p_{\sigma,\eta,+}\frac{|\nabla A|^2}{H^2}\,d\mu\,.$$

Combining these estimates we arrive at

$$\frac{d}{dt}\int G^p_{\sigma,\eta,+}\,d\mu \le -\left(p^2-p^{3/2}-p-\frac{1}{\gamma_2}\sigma p^{5/2}\right)\int G^{p-2}_{\sigma,\eta,+}|\nabla G_{\sigma,\eta}|^2\,d\mu$$
$$-\left(\gamma_1 p - Cp^{1/2} - \frac{1}{\gamma_2}\sigma(p^{3/2}+p)\right)\int G^p_{\sigma,\eta,+}\frac{|\nabla A|^2}{H^2}\,d\mu\ .$$

where $C = C(n)$. Therefore if we choose p large and $\sigma \approx p^{-1/2}$ we see that the right hand side is non-positive. This yields the following proposition.

Proposition 5.0.8. *There exists and $l = l(n, \eta)$ such that*

$$\frac{d}{dt} \int G^p_{\sigma,\eta,+} \, d\mu \leq 0$$

if $p \geq l^{-1}, \sigma \leq l/\sqrt{p}$.

From the L^p-estimate of the previous Proposition one can derive a uniform bound on the supremum of $G_{\sigma,\eta}$ with the procedure of [24, Theorem 5.1]. Let

$$k_0 := \sup_{\sigma \in [0,1]} \sup_{M_0} G_{\sigma,\eta}$$

and set for $k \geq k_0$

$$v = (G_{\sigma,\eta} - k)^{p/2}_+ , \qquad A(k,t) = \{x \in M \,|\, v(x,t) > 0\} \ .$$

Computing as before, see (5.7), we obtain for p large enough,

$$\frac{d}{dt} \int v^2 \, d\mu + \int |\nabla v|^2 \, d\mu \leq C_0 \sigma p \int_{A(k,t)} G^p_{\sigma,\eta} H^2 \, d\mu \ . \tag{5.8}$$

Note that the term on the right hand side arises since we estimate, using that we have the bound $|A|^2 \leq C_0 H^2$,

$$G_{\sigma,\eta}(G_{\sigma,\eta} - k)^{p-1}_+ |A|^2 \leq C_0 G^p_{\sigma,\eta} H^2 \ .$$

We now need the Michael-Simon Sobolev inequality.

Theorem 5.0.9 ([32]). *Assume $F : M^n \to \mathbb{R}^{n+1}$ is a smooth immersion. Then there exits a constant C, depending only on n, such that*

$$\left(\int |h|^{\frac{n}{n-1}} \, d\mu \right)^{\frac{n-1}{n}} \leq C \int |\nabla h| + |h||H| \, d\mu$$

for any $h \in C^{0,1}_c(M)$.

Choosing $q = n/(n-1) > 1$ (note that we assume $n \geq 3$), and using Hölder's inequality this implies

$$\left(\int v^{2q}\, d\mu\right)^{1/q} \leq C \int |\nabla v|^2 + C\left(\int_{A(k,t)} H^n\, d\mu\right)^{2/n}\left(\int v^{2q}\, d\mu\right)^{1/q}. \tag{5.9}$$

Now note that

$$H^n G^p_{\sigma,\eta} = (H^{n/p} G_{\sigma,\eta})^p = G^p_{\sigma',\eta}$$

where $\sigma' = \sigma + \frac{n}{p}$ and thus

$$\int\limits_{M_t} H^n G^p_{\sigma,\eta}\, d\mu = \int G^p_{\sigma',\eta}\, d\mu\ .$$

We assume that $\sigma \leq \frac{l}{2\sqrt{p}}$ and $p \geq \max\{1/l, 4n^2/l^2\}$ where l is given as in Proposition 5.0.8 and thus

$$\sigma' = \sigma + \frac{n}{p} \leq \frac{l}{2\sqrt{p}} + \frac{n}{\sqrt{p}}\frac{1}{\sqrt{p}} \leq \frac{l}{\sqrt{p}}.$$

Thus Proposition 5.0.8 yields

$$\begin{aligned}\left(\int_{A(k,t)} H^n\, d\mu\right)^{2/n} &\leq k^{-2p/n}\left(\int_{A(k,t)} H^n G^p_{\sigma,\eta}\, d\mu\right)^{2/n}\\ &\leq k^{-2p/n}\left(\int_{M_0} G^p_{\sigma',\eta}\, d\mu\right)^{2/n}\\ &\leq \left(\frac{(1+|M_0|)k_0}{k}\right)^{2p/n}.\end{aligned} \tag{5.10}$$

Thus we can fix $k_1 > k_0$ large enough such that, for any $k \geq k_1$ we may absorb the last term in (5.9)and then exploit the $|\nabla v|$ term in (5.9) to obtain

$$\frac{d}{dt}\int v^2\, d\mu + \frac{1}{C_1}\left(\int v^{2q}\, d\mu\right)^{1/q} \leq C_0 \sigma p \int_{A(k,t)} G^p_{\sigma,\eta} H^2\, d\mu\,. \tag{5.11}$$

Note that since $\int_{M_0} v^2\,d\mu = 0$ this yields, integrating over $[0,T)$ that

$$\sup_{[0,T]} \int\limits_{A(k,t)} v^2\,d\mu + \frac{1}{C_1}\int\limits_0^T \left(\int v^{2q}\,d\mu\right)^{1/q} dt \le C_0\sigma p \int\limits_0^T \int\limits_{A(k,t)} G^p_{\sigma,\eta} H^2\,d\mu\,dt \tag{5.12}$$

Now we use interpolation inequalities for L^p-spaces

$$\left(\int_{A(k,t)} v^{2q_0}\,d\mu\right)^{1/q_0} \le \left(\int_{A(k,t)} v^{2q}\,d\mu\right)^{a/q} \left(\int_{A(k,t)} v^2\,d\mu\right)^{(1-a)},$$

where $1/q_0 = a/q + (1-a)$ with $a = 1/q_0$ such that $1 < q_0 < q$. Then we have, denoting the right hand side of (5.12) with R

$$\begin{aligned}
\int\limits_0^T \int_{A(k,t)} v^{2q_0}\,d\mu\,dt &\le \int\limits_0^T \left(\int_{A(k,t)} v^{2q}\,d\mu\right)^{1/q} \left(\int_{A(k,t)} v^2\,d\mu\right)^{(q_0-1)} dt \\
&\le R^{q_0-1}\int\limits_0^T \left(\int_{A(k,t)} v^{2q}\,d\mu\right)^{1/q} dt \\
&\le C_1 R^{q_0-1} R = C_1 R^{q_0}
\end{aligned}$$

This yields, assuming w.l.o.g that $C_1 \ge 1$ that

$$\begin{aligned}
\left(\int\limits_0^T \int_{A(k,t)} v^{2q_0}\,d\mu\,dt\right)^{1/q_0} &\le C_2\sigma p \int\limits_0^T \int_{A(k,t)} G^p_{\sigma,\eta} H^2\,d\mu\,dt \\
&\le C_2\sigma p \|A(k)\|^{1-1/r} \left(\int\limits_0^T \int_{A(k,t)} G^{pr}_{\sigma,\eta} H^{2r}\,d\mu\,dt\right)^{1/r}
\end{aligned}$$

where $r > 1$ is to be chosen and

$$\|A(k)\| = \int\limits_0^T \int\limits_{A(k,t)} d\mu\,dt\ .$$

Again using Hölder's inequality we obtain

$$\int_0^T \int_{A(k,t)} v^p \, d\mu \, dt \le C_2 \sigma p \|A(k)\|^{1+b-1/r} \left(\int_0^T \int_{A(k,t)} G_{\sigma,\eta}^{pr} H^{2r} \, d\mu \, dt \right)^{1/r}$$

where $b = (q-1)/(2q-1)$. We now choose r large enough such that $\gamma := 1 + b - 1/r > 1$. With an argument as in (5.10) we can estimate the second factor on the right hand side provided p, σ^{-1} are larger than suitable constants depending only on n, η. We fix σ and p accordingly. Thus there is a constant C_3 such that, for all $h > k \ge k_1$,

$$|h-k|^p \|A(h)\| \le \int_0^T \int_{A(k,t)} v^p \, d\mu \, dt \le C_3^p \sigma p \|A(k)\|^{\gamma} \, .$$

By Stampacchia iteration [36, Lemma 4.1] we can conclude that

$$\|A(k,t)\| = 0 \qquad \forall\, k > k_1 + d^{1/p}$$

where

$$d = C_3^p \sigma 2^{p\gamma/(\gamma-1)} \|A(k_1)\|^{\gamma-1} \, .$$

Note that $\|A(k_1)\| \le T|M_0|$. Note that by the avoidance principle T can be bounded by a constant C_4 depending only on M_0. This yields the uniform bound

$$|A| \le \frac{1}{\sqrt{n-1}} H + \eta H + C_5 H^{1-\sigma}$$

where $C_5 = C_5(M_0, n, \eta)$. Squaring this inequality and using Young's inequality we arrive at the following theorem, compare [30, Theorem 5.3].

Theorem 5.0.10. *Let $(M_t)_{t\in[0,T)}$ be a closed two-convex solution to mean curvature flow for $n \ge 3$. Then for any $\eta > 0$ there exists $C_\eta = C(\eta, M_0)$ such that*

$$|A|^2 - \frac{1}{n-1} H^2 \le \eta H^2 + C_\eta$$

on M_t for any $t \in [0,T)$.

Recalling the identity

$$|A|^2 - \frac{1}{n-1}H^2 = \frac{1}{n-1}\left(\sum_{1<i<j\leq n}(\lambda_i - \lambda_j)^2 + \lambda_1(n\lambda_1 - 2H)\right) ,$$

this implies the following *cylindrical estimate*:

Corollary 5.0.11. *Let $(M_t)_{t\in[0,T)}$ be a closed two-convex solution to mean curvature flow for $n \geq 3$. Then for any $\eta > 0$ there exists $C_\eta = C(\eta, M_0)$ such that*

$$|\lambda_1| \leq \eta H \Longrightarrow |\lambda_j - \lambda_k| \leq c\eta H + C_\eta, \qquad j,k > 1$$

on M_t for any $t \in [0,T)$, where c only depends on n.

Assuming that $\lambda_1 \leq 0$ we see that the identity also implies that

$$|\lambda_1|(n|\lambda_1| + 2H) \leq \eta H^2 + C_\eta$$

which yields the *convexity estimate* of Huisken-Sinestrari [28]:

Corollary 5.0.12. *Let $(M_t)_{t\in[0,T)}$ be a closed two-convex solution to mean curvature flow for $n \geq 3$. Then for any $\eta > 0$ there exists $C_\eta = C(\eta, M_0)$ such that*

$$\lambda_1 \geq -\eta H - C_\eta$$

on M_t for any $t \in [0,T)$.

From this estimate one can obtain an estimate for the gradient of the curvature. Compared to other gradient estimates for mean curvature available in the literature, see for example [10, 14], this one is a pointwise estimate and does not depend on the maximum of the curvature in a suitable neighbourhood. This is especially helpful when considering blow-ups. A similar estimate for Ricci flow has been obtained by Perelman [33, 34] by a completely different approach.

Theorem 5.0.13 (Huisken-Sinestrari)**.** *Let $(M_t)_{t\in[0,T)}$ be a closed two-convex solution to mean curvature flow for $n \geq 3$. Then there exits a constant $\gamma_1 = \gamma_1(n)$ and a constant $\gamma_2 = \gamma_2(n)$ such that along the flow the uniform estimate*

$$|\nabla A|^2 \leq \gamma_2 |A|^4 + \gamma_3$$

holds for all $t \in [0, T)$.

Proof. The proof follows from the maximum principle applied to a suitable test-function. An important tool is the following inequality, see [24, Lemma 2.1], valid on any immersed hypersurface,

$$|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2 \,. \tag{5.13}$$

Observe that $\frac{3}{n+2} > \frac{1}{n-1}$ if $n \geq 3$. Let us set

$$\kappa_n = \frac{1}{2}\left(\frac{3}{n+2} - \frac{1}{n-1}\right) \,. \tag{5.14}$$

By Theorem 5.0.10 there exists $C_0 > 0$ such that

$$\left(\frac{1}{n-1} + \kappa_n\right) H^2 - |A|^2 + C_0 \geq 0 \,.$$

We define

$$g_1 := \left(\frac{1}{n-1} + \kappa_n\right) H^2 - |A|^2 + 2C_0 \,, \qquad g_2 = \frac{3}{n+2}H^2 - |A|^2 + 2C_0.$$

Then we have $g_2 \geq g_1 \geq C_0$ and so $g_1 - 2C_0 = 2(g_i - C_0) - g_i \geq -g_i$ for $i = 1, 2$. Using the evolution equations for $|A|^2, H^2$ and the inequality (5.13) we get

$$\begin{aligned}
\frac{\partial}{\partial t}g_1 - \Delta g_1 &= -2\left(\left(\frac{1}{n-1} + \kappa_n\right)|\nabla H|^2 - |\nabla A|^2\right) + 2|A|^2(g_1 - 2C_0) \\
&\geq 2\left(1 - \frac{n+2}{3}\left(\frac{1}{n-1} + \kappa_n\right)\right)|\nabla A|^2 - 2|A|^2 g_1 \\
&= 2\kappa_n \frac{n+2}{3}|\nabla A|^2 - 2|A|^2 g_1 \,.
\end{aligned} \tag{5.15}$$

Similarly

$$\frac{\partial}{\partial t}g_2 - \Delta g_2 = -2\left(\frac{3}{n+2}|\nabla H|^2 - |\nabla A|^2\right) + 2|A|^2(g_2 - 2C_0) \geq -2|A|^2 g_2 \tag{5.16}$$

In addition we have, see Theorem 2.1.4,

$$\frac{\partial}{\partial t}|\nabla A|^2 - \Delta|\nabla A|^2 \leq -2|\nabla^2 A|^2 + c_n|A|^2|\nabla A|^2 \tag{5.17}$$

for a constant c_n depending only on n. Using these equations one can show directly that the following inequality holds

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \Delta\left(\frac{|\nabla A|^2}{g_1 g_2}\right) \\
&\quad \leq \frac{2}{g_2}\left\langle \nabla g_2, \nabla \frac{|\nabla A|^2}{g_1 g_2}\right\rangle + \frac{|A|^2|\nabla A|^2}{g_1 g_2}\left((c_n+4) - 2\kappa_n^2 \frac{n+2}{3n}\frac{|\nabla A|^2}{g_1 g_2}\right) .
\end{aligned}$$

Thus we get a contradiction if at a new maximum we have

$$\frac{|\nabla A|^2}{g_1 g_2} > \frac{3n(c_n+4)}{2\kappa_n^2(n+2)} .$$

This yields the desired statement. $\square$

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