Partial Differential Equations II

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Felix Schulze

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1 Preface

These are lecture notes for a course in 'Partial Differential Equations II' given at the Free University Berlin in the winter term 2011/2012.

I do not claim originality for any of the results in these notes. As sources I have freely used the wonderful books of Evans [2] and Gilbarg-Trudinger [3] as well as lecture notes from a course given by G. Huisken in Tübingen. I have especially used these lecture notes to give a simplified version of the 'Schauder estimates by scaling' due to L. Simon, see [4]. To my knowledge this simplified version goes back to lectures given by L. Simon.

I'd be grateful for letting me know of any mistakes or typos one might find in these notes.

London, June 2019

2 Overview

2.1 Schauder estimates

We will study the theory for linear elliptic equations of second order of the form

$$(\star) \qquad \begin{aligned} Lu := a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u &= f(x) \quad \text{in } \Omega \subset \mathbb{R}^n \\ u &= \varphi \quad \text{on } \partial\Omega\,. \end{aligned}$$

We know from PDE I that if L can be written in divergence form (this depends on the regularity of the a^{ij}), then under suitable conditions there exists a solution to the above problem in Sobolev-spaces. Question: When do solutions exist in $C^2, C^3, \dots$? As an example, we will prove the following:

If $c \leq 0,\ f \in C^{0,\alpha}(\bar{\Omega}), \varphi \in C^{2,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$ and suitable regularity of the coefficients a^{ij}, b^i, c, then there exists a unique solution to $(\star)$ in $C^{2,\alpha}(\bar{\Omega})$.

To show this result we will need a-priori estimates for u, Du, D^2u. An example are the following interior Schauder estimates:

Let $u \in C^{2,\alpha}(\Omega)$ be a solution to $(\star)$ and $\|a^{ij}, b^i, c\|_{C^{0,\alpha}(\Omega)} \leq C$. Then for $\Omega' \Subset \Omega$ it holds

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C'(C, \Omega, \Omega') \big(\sup_{\Omega} |u| + \|f\|_{C^{0,\alpha}(\Omega)} \big)\,.$$

2.2 Existence theory for quasilinear equations of second order

We want to solve elliptic equations of second order, when the coefficients are allowed to depend on the solution u, for $u \in C^2(\Omega)$, and its gradient Du:

$$(\star\star) \qquad \begin{aligned} Qu := a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du) &= f(x) \quad \text{in } \Omega \subset \mathbb{R}^n \\ u &= \varphi \quad \text{on } \partial\Omega\,. \end{aligned}$$

We know from PDE I that there is a maximum principle for equations of this form. So when do we get existence and uniqueness of solutions to $(\star\star)$?

A central example will be the minimal surface equation. Let $M = \text{graph}(u),\ u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$ be a minimal surface, then u solves

$$\begin{aligned} 0 = \mathcal{A}(u) &:= \text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right) D_{ij} u \\ &= a^{ij}(Du) D_{ij} u\ . \end{aligned}$$

We see that for any $\xi \in \mathbb{R}^n$ it holds

$$a^{ij}(Du)\xi_i\xi_j \geq \frac{1}{(1+|Du|^2)^{3/2}}|\xi|^2$$

and thus $\mathcal{A}$ is uniformly elliptic, provided $|Du|$ is uniformly bounded.

To show the existence of solutions to equations of the type $(\star\star)$ it turns out that we need in addition to the Schauder estimates further estimates for linear elliptic equations in divergence form

$$(+) \qquad \tilde{L}u := D_i\big(a^{ij}(x)D_j u + b^i(x)u\big) + c^i(x)D_i u + d(x)u = f \qquad \text{in } \Omega \subset \mathbb{R}^n.$$

The so-called Nash-Moser-DeGiorgi estimates show Hölder-regularity of weak solutions:

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $(+)$, $\Omega' \Subset \Omega$, then for $q > n$ it holds that

$$\|u\|_{C^{0,\alpha}(\bar{\Omega}')} \leq C(\Omega, \Omega', q, \text{coeff.}) \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^{q/2}(\Omega)}\right)$$

for some $0 < \alpha < 1$.

Using these estimates we will, as a guiding example, show under suitable conditions the existence of surfaces with prescribed mean curvature. That is we will prove existence of solutions to the following problem

$$\begin{aligned} \mathcal{A}(u) &= \mathcal{H}(x,u) \qquad && \text{in } \Omega \subset \mathbb{R}^n \\ u &= \psi && \text{on } \partial\Omega\,, \end{aligned}$$

where $\mathcal{H} : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a given function.

References:

Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, AMS, 1998.

David Gilbarg, Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer 2001.

3 Schauder theory

3.1 First a-priori estimates

In the following we will always assume that $\Omega \subset \mathbb{R}^n$ is a domain, that is open, bounded and connected. We consider operators of the form

$$Lu := a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \tag{3.1}$$

with $a^{ij}, b^i, c : \Omega \to \mathbb{R}$ bounded, that is $|a^{ij}|, |b^i|, |c| \leq M$. Furthermore the matrix (a^{ij}) is supposed to be symmetric and uniformly elliptic, i.e. $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \ \forall x \in \Omega, \xi \in \mathbb{R}^n$ and a fixed $\lambda > 0$.

We recall the following theorems.

Theorem 3.1.1 (Strong maximum principle)**.** *Consider L as in* (3.1) *with $c \leq 0$. If $u \in C^2(\Omega)$ fulfills $Lu \geq 0$, then u cannot achieve a positive maximum in Ω unless u is constant.*

Corollary 3.1.2. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solve $Lu = 0$ on Ω with $c \equiv 0$. Then*

$$\inf_{\partial\Omega} u \leq u \leq \sup_{\partial\Omega} u \ .$$

Corollary 3.1.3. *Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ fulfill $Lu \leq 0$ and $Lv \geq 0$ on Ω with $c \leq 0$. Then*

$$u - v \geq \min\{0, \inf_{\partial\Omega}(u - v)\} \ .$$

With the help of the maximum principle it is relatively easy to show an a-priori sup-estimate for u.

Theorem 3.1.4. *Let $Lu \geq f \ (= f), c \leq 0, u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

$$\sup_{\Omega} u \, (|u|) \leq \sup_{\partial\Omega} u^+ \, (|u|) + C \sup_{\Omega} \frac{|f^-|}{\lambda} \left(\frac{|f|}{\lambda}\right)$$

with $C = C\big(\operatorname{diam}\Omega, \sup_{\Omega} \left(|b|/\lambda\right)\big)$.

Proof. W.l.o.g. we can assume that $\Omega \subset \{x \in \mathbb{R}^n | 0 < x_1 < d\}$. Let $\beta := \sup_\Omega |b^1|/\lambda$ and define

$$L_0 u := a^{ij} D_{ij} u + b^i D_i u \ .$$

For any $\alpha \geq \beta + 1$ we have

$$L_0 e^{\alpha x_1} = (\alpha^2 a^{11} + \alpha b^1) e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta) e^{\alpha x_1} \geq \alpha\lambda \geq \lambda \ .$$

We then take

$$v := \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_\Omega \left(\frac{|f^-|}{\lambda} \right)$$

which is non-negative on Ω. Since

$$Lv = L_0 v + cv \leq -\lambda \sup_\Omega \left(\frac{|f^-|}{\lambda} \right) = -\sup_\Omega (|f^-|)$$

it follows that

$$L(v-u) \leq -(\sup_\Omega |f^-| + f) \leq 0$$

on Ω. Since $v - u \geq 0$ on $\partial\Omega$, we have by the maximum principle $u \leq v$ and thus

$$\sup_\Omega u \leq \sup_\Omega v \leq \sup_{\partial\Omega} u^+ + \frac{C}{\lambda} \sup_\Omega |f^-| \ ,$$

where we chose $C = e^{\alpha d} - 1$ and $\alpha = \beta + 1$. For the statement in the case of $Lu = f$ we use that then $L(-u) \geq -f$. □

Corollary 3.1.5. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solve $Lu = f$, $u|_{\partial\Omega} = \varphi$, with $c \leq 0$. Then the solution is unique and depends continuously on the data.*

Proof. If we have $Lu_1 = f_1$, $Lu_2 = f_2$ and $u_1|_{\partial\Omega} = \varphi_1$, $u_2|_{\partial\Omega} = \varphi_2$, then

$$L(u_1 - u_2) = f_1 - f_2 \ \text{ with } \ (u_1 - u_2)\Big|_{\partial\Omega} = \varphi_1 - \varphi_2 \ .$$

By Theorem 3.1.4 it follows that

$$\sup_\Omega |u_1 - u_2| \leq \sup_{\partial\Omega} |\varphi_1 - \varphi_2| + C \sup_\Omega |f_1 - f_2| \ .$$

□

In the rest of this section we want to show that the maximum principle allows us as well to conclude a-priori bounds for the norm of the gradient $|Du|$ for solutions of $Lu = f$. This is done in two steps:

i) Find an estimate for $\sup_{\partial\Omega} |Du|$ - which may depend on $\varphi, D\varphi, \sup_\Omega |u|, f$. Since $u = \varphi$ on $\partial\Omega$ it suffices to estimate $\left|\frac{\partial u}{\partial \nu}\right|$. We will do this by constructing upper and lower barriers.

ii) We will show that there is a maximum principle for $|Du| \rightsquigarrow$ we can estimate $\sup_\Omega |Du|$ by $\sup_{\partial\Omega} |Du|$, $\sup_\Omega |u|$, $\sup_\Omega |f|$ and $\sup_\Omega |Df|$.

We will in the following assume that the domain Ω satisfies an uniform exterior sphere condition. That there exists a $R > 0$, such that for all $x_0 \in \partial\Omega$ there exists a ball $B_R(y)$ such that $\bar{\Omega} \cap \bar{B}_R(y) = \{x_0\}$. Note that any domain Ω with boundary $\partial\Omega \in C^2$ (i.e. $\partial\Omega$ is a embedded hypersurface in C^2) fulfills such a condition for some $R > 0$ small enough.

Lemma 3.1.6. *Let $B_R(y)\mathbb{R}^n, y \in \mathbb{R}^n$ arbitrary. Let $d(x) := \operatorname{dist}\big(x, \partial B_R(y)\big)$ for $x \in \big(B_R(y)\big)^c$. Then $d \in C^\infty(\mathbb{R}^n \setminus \bar{B}_R(y))$, $|Dd| = 1$ and*

$$|D^2 d| \leq \frac{C(n)}{R}.$$

Proof. We can assume $y = 0$. Then $d(x) = |x| - R$ and one can show the above statements by direct computations. □

Theorem 3.1.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega \in C^2$. Let $\varphi \in C^2(\bar{\Omega})$ and assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega\,, \\ u &= \varphi \qquad \text{on } \partial\Omega \end{aligned}$$

with $c \leq 0$. Then it holds that

$$\sup_{\partial\Omega} |Du| \leq C(\partial\Omega, \sup_\Omega |f|, \|\varphi\|_{C^2(\Omega)}, L)\ .$$

Proof. For $x_0 \in \partial\Omega$ we can split

$$Du(x_0) = (Du(x_0))^T + (Du(x_0))^\perp = (D\varphi(x_0))^T + \frac{\partial u}{\partial \nu}(x_0)\nu$$

where ν is the outward unit normal to $\partial\Omega$. Thus it suffices to estimate $\left|\frac{\partial u}{\partial \nu}\right|$ to get an estimate of $|Du|$ on the boundary. To do this we construct barriers $\delta^+, \delta^- \in C^2(\bar{\Omega})$ with

(i) $$L\delta^+ \leq f\,,\ L\delta^- \geq f \quad \text{on } \Omega$$

(ii) $$\delta^+ \geq \varphi\,,\quad \delta^- \leq \varphi \quad \text{on } \partial\Omega$$

(iii) $$\delta^-(x_0) = u(x_0) = \varphi(x_0) = \delta^+(x_0)\,.$$

Assuming we have found such barriers we get by the maximum principle that

$$\delta^-(x) \le u(x) \le \delta^+(x) \quad \forall\, x \in \bar{\Omega}$$

and thus

$$\delta^-(x) - \delta^-(x_0) \le u(x) - u(x_0) \le \delta^+(x) - \delta^+(x_0)\ ,$$

which implies

$$\Big|\frac{\partial u}{\partial \nu}\Big| \le \max\Big(\Big|\frac{\partial \delta^+}{\partial \nu}\Big|(x_0), \Big|\frac{\partial \delta^+}{\partial \nu}\Big|(x_0)\Big) \le \sup_{\Omega}\big(|D\delta^+|, |D\delta^-|\big)\ .$$

Construction of δ^+:

Let $x_0 \in \partial\Omega$ as above, $B_R(y_0) \subset \Omega^c$ such that $\bar{\Omega} \cap \bar{B}_R(y_0) = \{x_0\}$. For $d(x) := \mathrm{dist}(x, \partial B_R(y_0))$, and $\alpha, \gamma > 0$ define

$$\delta^{\pm}(x) := \varphi(x) \pm \frac{\gamma}{\alpha}\big(1 - e^{-\alpha d(x)}\big)\ .$$

It is clear that $\delta^{\pm} \in C^2(\bar{\Omega})$ and that (iii) is fulfilled. Since $d \ge 0$ on Ω we have $(1 - e^{-\alpha d(x)}) \ge 0$ and thus

$$\delta^+ \ge \varphi \ \text{ and } \ \delta^- \le \varphi\ ,$$

which implies that also (ii) is fulfilled. To see that (i) is true as well we compute

$$D_i\delta^+ = D_i\varphi + \gamma e^{-\alpha d} D_i d\ , \quad D_{ij}\delta^+ = D_{ij}\varphi - \alpha\gamma e^{-\alpha d} D_i d D_j d + \gamma e^{-\alpha d} D_{ij} d\,,$$

which gives

$$\begin{aligned} L\delta^+ &= a^{ij} D_{ij}\delta^+ + b^i D_i \delta^+ + c\delta^+ \\ &= a^{ij} D_{ij}\varphi - \alpha\gamma e^{-\alpha d} a^{ij} D_i d D_j d + \gamma e^{-\alpha d} a^{ij} D_{ij} d + b^i D_i \varphi \\ &\quad + \gamma e^{-\alpha d} b^i D_i d + c\varphi + c\frac{\gamma}{\alpha}(1 - e^{-\alpha d}) \\ &\le \gamma(-\alpha\lambda + C(1 + \frac{1}{R}))e^{-\alpha d} + C\|\varphi\|_{C^2(\Omega)}\,. \end{aligned}$$

Choose α big enough such that $\alpha\lambda \ge C(1 + \frac{1}{R}) + 1$ and define $C' := C\|\varphi\|_{C^2(\Omega)}$. This implies

$$L\delta^+ \le -\gamma e^{-\alpha d} + C'\ .$$

Since $d \le \mathrm{diam}(\Omega)$ we can choose γ big enough such that if

$$\gamma e^{-\alpha\,\mathrm{diam}(\Omega)} \ge C' + \sup_{\Omega} |f|$$

we obtain

$$L\delta^+ \le -\sup_{\Omega} |f| \le f\ ,$$

which is (i). The computation for δ^- is analogous. □

Remark 3.1.8: If $\partial\Omega \in C^2$ and $\varphi \in C^2(\partial\Omega)$ is given, then φ can be extended as a C^2-function $\tilde{\varphi}$ on Ω such that

$$\|\tilde{\varphi}\|_{C^2(\bar{\Omega})} \leq C(\partial\Omega, \|\varphi\|_{C^2(\partial\Omega)})\,.$$

The idea to get global gradient bounds is that if u fulfills an elliptic equation then also $|Du|^2$. As an easy example assume that $u \in C^3(\Omega) \cap C^1(\Omega)$ satisfies

$$\Delta u = 0\,.$$

We can differentiate this equation once in direction e_i and multiply with $D^i u$ to see that

$$D^i u\, \Delta D_i u = 0\,.$$

This then gives

$$\begin{aligned}\Delta|Du|^2 &= D^k D_k(D^i u D_i u) = 2D^k((D_k D^i u) D_i u) = 2D_i u \Delta D^i u + 2D_k D^i u D^k D_i u \\ &= 2|D^2 u|^2 \geq 0\end{aligned}$$

and thus

$$\sup_\Omega |Du|^2 \leq \sup_{\partial\Omega} |Du|^2\,.$$

This idea also works in the general case:

Theorem 3.1.9. *Let $\partial\Omega \in C^1$ and $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ be a solution of $Lu = f$, where $\|a^{ij}\|_{C^1(\bar{\Omega})}, \|b^i\|_{C^1(\bar{\Omega})}, \|c\|_{C^1(\bar{\Omega})} \leq M$ and $f \in C^1(\bar{\Omega})$. Then it holds that*

$$\sup_\Omega |Du| \leq \sup_{\partial\Omega} |Du| + C$$

where $C = C(\lambda, M, \sup_\Omega |u|, \|f\|_{C^1(\bar{\Omega})})$.

Remark 3.1.10: It is <u>not</u> required that $c \leq 0$. This makes it possible to apply this theorem to eigenfunctions satisfying:

$$a^{ij} D_{ij} u + b^i D_i u = -\sigma u$$

for some $\sigma =$ const. > 0 and $u|_{\partial\Omega} = 0$. Note that there is no a-priori estimate for $\sup_\Omega |u|$ which only depends on the data a^{ij}, b^i, since αu is again an eigenfunction for any $\alpha > 0$. Sup-estimates can only come from an extra condition like $\int_\Omega |u|^2\, dx = 1$.

Proof of Theorem 3.1.9. Differentiating $Lu = f$ in direction e_k we obtain that

$$a^{ij} D_{ij}(D_k u) + b^i D_i(D_k u) + c\, D_k u = D_k f - D_k a^{ij} D_{ij} u - D_k b^i D_i u - D_k c\, u\,.$$

Multiplying with $D^k u$ and summing over k gives

$$a^{ij}(D_{ij}(D_k u)D^k u) + b^i(D_i(D_k u)D^k u) + c|Du|^2 \geq -|Du|\|f\|_{C^1(\Omega)} - M(|D^2u||Du| + |Du|^2 + |u||Du|)\,.$$

Note that $D_i|Du|^2 = 2D^k u D_i(D_k u)$ and

$$D_{ij}|Du|^2 = 2D^k u D_{ij} D_k u + 2D_j D^k u D_i D_k u\ .$$

So we get

$$a^{ij}D_{ij}|Du|^2 - 2a^{ij}D_i D^k u D_j D_k u + b^i D_i|Du|^2 \geq -C_0(|D^2u||Du| + |Du|^2 + |Du|)$$

with $C_0 = C_0(M, \|f\|_{C^1(\Omega)}, \sup_\Omega |u|)$. Note that by the ellipticity we have

$$a^{ij}D_i D^k u D_j D_k u \geq \lambda |D^2 u|^2\,,$$

and thus

$$L_0(|Du|^2) = a^{ij}D_{ij}|Du|^2 + b^i D_i|Du|^2 \geq 2\lambda|D^2u|^2 - C_0(|D^2u||Du| + |Du|^2 + |Du|)\,.$$

By Young's inequality we can estimate

$$C_0|D^2u||Du| \leq \lambda|D^2u|^2 + \frac{C_0^2}{4\lambda}|Du|^2 \quad \text{and} \quad |Du| \leq |Du|^2 + \frac{1}{4}\,,$$

which gives

$$L_0(|Du|^2) \geq -C_1|Du|^2 - C_2$$

To compensate the first term on the RHS we use the a-priori estimate for $|u|$:

$$\begin{aligned} L_0(|u|^2) &= 2ua^{ij}D_{ij}u + 2ub^i D_i u + 2a^{ij}D_i u D_j u \geq 2u(f - cu) + 2\lambda|Du|^2 \\ &\geq 2\lambda|Du|^2 - C_3 \end{aligned}$$

Setting $g := |Du|^2 + \alpha u^2$ we obtain for $\alpha = (C_1 + 1)/(2\lambda)$

$$L_0(g) \geq (2\lambda\alpha - C_1)|Du|^2 - C_4 \geq |Du|^2 - C_4 \geq g - C_5$$

which implies that $\tilde{g} := g - C_5$ satisfies

$$L_0(\tilde{g}) - \tilde{g} \geq 0\,.$$

The maximum principle thus implies $\sup_\Omega \tilde{g} \leq \max\{\sup_{\partial\Omega} \tilde{g}, 0\}$, and thus

$$\sup_\Omega |Du|^2 \leq \sup_\Omega \tilde{g} + C_5 \leq \max\{\sup_{\partial\Omega} \tilde{g}, 0\} + C_5 \leq \sup_{\partial\Omega} |Du|^2 + \alpha \sup_{\partial\Omega} |u|^2 + C_5\,.$$

□

3.2 Local Schauder estimates

In this section we will prove local Schauder estimates. The presentation follows [4] and [3]. In the following let again $\Omega \subset \mathbb{R}^n$ be domain and $u \in C^2(\Omega)$ a solution of

$$(\star) \qquad Lu = a^{ij}D_{ij}u + B^i D_i u + c\,u = \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta} D^{\alpha+\beta} u = f$$

or $u \in C^1(\Omega)$ be a weak solution of

$$(\star\star) \qquad \begin{aligned} \tilde{L}u &= \sum_{|\alpha|,|\beta|\le 1} D^\alpha (a_{\alpha\beta} D^\beta u) = D_i(a^{ij}D_j u + b^i u) + c^i D_i u + du \\ &= D_i f^i + g = \sum_{|\beta|\le 1} D^\beta f_\beta \,, \end{aligned}$$

that is

$$\int_\Omega (a^{ij}D_j u + b^i u) D_j \varphi \, dx - \int_\Omega (c^i D_i u + du)\varphi \, dx = \int_\Omega f^i D_i \varphi - g\varphi \, dx$$

$\forall\, \varphi \in C^1_c(\Omega)$. We will study the continuity of $D^2 u$ in the case of $(\star)$ and the continuity of Du in the case of $(\star\star)$.

Hölder norms:

Let $h : \Omega \to \mathbb{R}$ and $A \subset \Omega$. For $\alpha \in (0, 1]$ we define the Hölder-seminorm by

$$[h]_{\alpha;\,A} := \sup_{\substack{x,y\in A\\ x\ne y}} \frac{|h(x) - h(y)|}{|x - y|^\alpha} \,.$$

We set

$$\begin{aligned} C^{0,\alpha}(\Omega) &:= \{h \in C^0(\Omega) \,|\, [h]_{\alpha;\,K} < \infty,\ \forall\, K \Subset \Omega\}, \\ C^{0,\alpha}(\bar\Omega) &:= \{h \in C^0(\bar\Omega) \,|\, [h]_{\alpha;\,\Omega} < \infty\}, \\ C^{k,\alpha}(\Omega) &:= \{h \in C^k(\Omega) \,|\, [D^\beta h]_{\alpha;\,K} < \infty,\ \forall\, K \Subset \Omega, \forall\, |\beta| = k\}, \\ C^{k,\alpha}(\bar\Omega) &:= \{h \in C^k(\bar\Omega) \,|\, [D^\beta h]_{\alpha;\,\Omega} < \infty,\ \forall\, |\beta| = k\}, \end{aligned}$$

and further seminorms by

$$\begin{aligned} [h]_{k,\,0;\,\Omega} &= |D^k h|_{0;\,\Omega} = \sup_{|\beta|=k} \sup_{\Omega} |D^\beta h|, \quad k = 0, 1, 2, \dots \\ [h]_{k,\,\alpha;\,\Omega} &= [D^k h]_{\alpha;\,\Omega} = \sup_{|\beta|=k} [D^\beta h]_{\alpha;\,\Omega} \,. \end{aligned}$$

With this seminorms, we can define the related norms

$$\|h\|_{C^k(\bar\Omega)} = |h|_{k;\,\Omega} = |h|_{k,\,0;\,\Omega} = \sum_{j=0}^{k}[h]_{j,\,0;\,\Omega} = \sum_{j=0}^{k}|D^j h|_{0;\,\Omega}\,,$$
$$\|h\|_{C^{k,\alpha}(\bar\Omega)} = |h|_{k,\,\alpha;\,\Omega} = |h|_{k;\,\Omega} + [h]_{k,\,\alpha;\,\Omega} = |h|_{k;\,\Omega} + [D^k h]_{\alpha;\,\Omega}\,.$$

We will in the following always assume that $L, \tilde L$ are uniformly elliptic, that is $\exists\,\lambda > 0$ s.t.

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall\, x\in\Omega,\ \forall\,\xi\in\mathbb{R}^n\,.$$

We will prove in this the follwoing two interior estimates

Theorem 3.2.1. *Let $u \in C^{1,\mu}(\bar B_R(x_0))$ be a weak solution to $(\star\star)$ and*

$$|a_{\alpha\beta}|_{0,\,\mu;\,B_R(x_0)} \leq \Lambda\,.$$

Then there exists for all $\theta\in(0,1)$ a constant $C = C(n,\theta,\lambda,\Lambda,\mu,R)$ such that

$$|u|_{1,\,\mu;\,B_{\theta R}(x_0)} \leq C(|u|_{0,\,B_R(x_0)} + \sum_{|\beta|\leq 1}|f_\beta|_{0,\,\mu;\,B_R(x_0)})\,.$$

Theorem 3.2.2. *Let $u \in C^{2,\mu}(\bar B_R(x_0))$ be a solution to $(\star)$ and*

$$|a_{\alpha\beta}|_{0,\,\mu;\,B_R(x_0)} \leq \Lambda\,.$$

Then there exists for all $\theta\in(0,1)$ a constant $C = C(n,\theta,\lambda,\Lambda,\mu,R)$ such that

$$|u|_{2,\,\mu;\,B_{\theta R}(x_0)} \leq C(|u|_{0,\,B_R(x_0)} + |f|_{0,\,\mu;\,B_R(x_0)})\,.$$

The strategy to prove these theorems is to first prove the estimates for second order elliptic operators with constant coefficients, and then use a perturbation argument.

Lemma 3.2.3. *Let $u\in C^2(\mathbb{R}^n)$ be a solution of $\bar a^{ij}D_{ij}u = f$, where the matrix $(\bar a^{ij})$ is constant, uniformly elliptic and satisfies $|\bar a^{ij}| < \Lambda$. If $[D^2u]_{\mu;\mathbb{R}^n} < \infty$ for some $0<\mu<1$, then*

$$[D^2u]_{\mu;\mathbb{R}^n} \leq C[f]_{\mu;\mathbb{R}^n}$$

with a constant $C = C(n,\lambda,\Lambda,\mu)$.

To prove this Lemma we will need the following interpolation inequality, which we will prove after the next proof of the previous lemma.

Lemma 3.2.4. *For all $\varepsilon > 0$ and integers $1\leq l\leq k$ there is a $C = C(\varepsilon,\mu,k,n)$, such that for all $h\in C^{k,\mu}(\bar B_R(x_0))$ it holds*

$$R^l|D^l h|_{0;\,B_R(x_0)} \leq \varepsilon R^{k+\mu}[D^k h]_{\mu;\,B_R(x_0)} + C|h|_{0;\,B_R(x_0)}\,.$$

Proof of Lemma 3.2.3. We will prove the statement by contradiction. So assume none such C exists. Then there are sequences $\bar{a}_m^{ij}$, f_m, u_m, such that the $\bar{a}_m^{ij}$ satisfy the conditions of the statement, $\bar{a}_m^{ij} D_{ij} u_m = f_m$ and

$$[f_m]_{\mu;\mathbb{R}^n} < \frac{1}{m}[D^2 u_m]_{\mu;\mathbb{R}^n} < \infty\,.$$

By the definition of the Hölder-seminorm it follows that there exist $x_m, y_m \in \mathbb{R}^n, x_m \neq y_m$ such that

$$|D^2 u_m(x_m) - D^2 u_m(y_m)| \geq \big(1 - \frac{1}{m}\big)[D^2 u_m]_{\mu;\mathbb{R}^n} \cdot |x_m - y_m|^\mu\,. \tag{3.2}$$

We will apply a rescaling argument as follows. Let $\sigma_m = |x_m - y_m|,\ \rho_m = [D^2 u_m]_{\mu;\mathbb{R}^n}$, and define

$$\tilde{u}_m(x) := \sigma_m^{-2-\mu}\rho_m^{-1} u_m(\sigma_m x + x_m)\,, \qquad \tilde{f}_m(x) := \sigma_m^{-\mu}\rho_m^{-1} f_m(\sigma_m x + x_m)\,,$$

then $\tilde{u}_m$ satisfies

$$\bar{a}_m^{ij} D_{ij}\tilde{u}_m = \tilde{f}_m\,.$$

And note that

$$\begin{aligned}
[D^2\tilde{u}_m]_{\mu;\mathbb{R}^n} &= \sup_{x\neq y}\frac{|D^2\tilde{u}_m(x) - D^2\tilde{u}_m(y)|}{|x-y|^\mu}\\
&= \sup_{x\neq y}\frac{\sigma_m^{-\mu}\rho_m^{-1}|D^2 u_m(\sigma_m x + x_m) - D^2 u_m(\sigma_m y + x_m)|}{\sigma^{-\mu}|\sigma_m(x-y)|^\mu}\\
&= \sup_{\tilde{x}\neq\tilde{y}}\frac{|D^2 u_m(\tilde{x}) - D^2 u_m(\tilde{y})|}{|\tilde{x}-\tilde{y}|^\mu}\cdot\frac{1}{\rho_m} = 1
\end{aligned}$$

Analogously one sees that $[\tilde{f}_m]_{\mu;\mathbb{R}^n} \leq \frac{1}{m}$ and (3.2) gives

$$|D^2\tilde{u}_m(\eta_m) - D^2\tilde{u}_m(0)| \geq 1 - \frac{1}{m}$$

for $\eta_m = \sigma_m^{-1}(y_m - x_m)$ and all $m \in \mathbb{N}$. Note that $|\eta_m| = 1$.

To proceed we would like to apply Arzela-Ascoli to extract a convergent subsequence to reach a contradiction. The problem is that we have no sup-bound for $\tilde{u}_m$! To circumvent this, let γ_m be the Taylor-polynomial of second order at $x_0 = 0$ for $\tilde{u}_m$, that is

$$\gamma_m(x) := \tilde{u}_m(0) + D_i\tilde{u}_m(0)x^i + \tfrac{1}{2}D_{ij}\tilde{u}_m(0)x^i x^j\,,$$

and we set $v_m := \tilde{u}_m - \gamma_m$. We can estimate, using the Taylor expansion for $\tilde{u}_m$ around 0,

$$\tilde{u}_m(x) = \tilde{u}_m(0) + D_i\tilde{u}_m(0)x^i + \tfrac{1}{2}D_{ij}\tilde{u}_m(\theta x)x^i x^j$$

for some $\theta \in (0,1)$ that

$$\begin{aligned}\sup_{B_R(0)} |v_m| &\le R^2 \sup_{B_R(0)} |D^2\tilde{u}_m(x) - D^2\tilde{u}_m(0)| \le R^{2+\mu}[D^2\tilde{u}]_{\mu;\, B_R(0)} \\ &\le R^{2+\mu}[D^2\tilde{u}]_{\mu;\, \mathbb{R}^n} \le R^{2+\mu}\,. \end{aligned} \tag{3.3}$$

We note that v_m again fulfills an elliptic equation:

$$\bar{a}^{ij}_m D_{ij} v_m = \bar{a}^{ij}_m D_{ij}\tilde{u}_m - \bar{a}^{ij} D_{ij}\gamma_m = \tilde{f}_m - \bar{a}^{ij}_m D_{ij}\tilde{u}_m(0) = \tilde{f}_m - \tilde{f}_m(0)\,,$$

and we have

$$[D^2 v_m]_{\mu;\, \mathbb{R}^n} = 1\,, \quad |D^2 v_m(\eta_m) - D^2 v_m(0)| = |D^2\tilde{u}_m(\eta_m) - D^2\tilde{u}_m(0)| \ge 1 - \frac{1}{m}\,.$$

We now apply the interpolation estimate from Lemma 3.2.4 with $k = 2$ and $l = 1, 2$ to v_m, and use (3.3):

$$\begin{aligned} |D^l v_m|_{0;\, B_R(0)} &\le \varepsilon R^{2-l+\mu}[D^2 v_m]_{\mu,\, B_R(0)} + C_\varepsilon R^{-l}|v_m|_{0,\, B_R(0)} \\ &\le \varepsilon R^{2-l+\mu} + C_\varepsilon R^{2-l+\mu} = C R^{2-l+\mu}\,. \end{aligned}$$

So by Arzela-Ascoli there is a subsequence $v_{m'}$ which converges locally uniformly in C^2 to a function $v \in C^{2,\mu}(\mathbb{R}^n)$ with

$$[D^2 v]_{\mu;\, \mathbb{R}^n} \le 1\,, \quad D^2 v(0) = 0\,, \quad |D^2 v(\eta)| = 1\,, \tag{3.4}$$

where η is an accumulation point of $(\eta_{m'})$. Note that $|\eta| = 1$. Let $\bar{a}^{ij}$ be an accumulation point of $(\bar{a}^{ij}_{m'})$, then v fulfills

$$\bar{a}^{ij} D_{ij} v = 0 \quad \text{on } \mathbb{R}^n\,,$$

since $[\tilde{f}_m]_{\mu, \mathbb{R}^n} \le 1/m \to 0$ and $\bar{a}^{ij}_m D_{ij} v_m = f_m - f_m(0)$. Note that $\bar{a}^{ij}$ is again uniformly elliptic, and satisfies $|\bar{a}^{ij}| \le \Lambda$. Since $\bar{a}^{ij}$ is constant we have

$$D^i(\bar{a}^{ij} D_j v) = \bar{a}^{ij} D_{ij} v = 0\,,$$

and so since the $\bar{a}^{ij}$ are smooth and this is an equation in divergence form, we have by PDE 1 that actually $v \in C^\infty(\mathbb{R}^n)$.

Claim:

$$\sup_{B_{R/2}(0)} |D^3 v| \le C(n, \lambda, \Lambda) R^{-3} \sup_{B_R(0)} |v|\,. \tag{3.5}$$

Applying this claim, this gives with (3.3) that

$$\sup_{B_{R/2}(0)} |D^3 v| \le C R^{\mu - 1}$$

with $C = C(n, \lambda, \Lambda, \mu)$. Since $\mu \in (0,1)$ this implies by letting $R \to \infty$ that $D^3 v \equiv 0$ and v is a polynomial of second order. This contradicts (3.4).

Proof of Claim (3.5)*:*
We first want to prove interior gradient bounds for u, where $u \in C^\infty(B_R(0))$ satisfies:

$$\bar{L}u = \bar{a}^{ij} D_{ij} v = 0$$

with $(\bar{a}^{ij})$ uniformly elliptic and $|\bar{a}^{ij}| \leq \Lambda$. We consider the testfunction

$$g := |Du|^2 (u^2 + k) \cdot \eta^2$$

for a constant $k > 0$ and

$$\eta(x) = (\tilde{R}^2 - |x|^2)^+ \, ,$$

for some $0 < \tilde{R} < R$. Since $\eta|_{B_R(0) \setminus B_{\tilde{R}}(0)} = 0$, we have that g attains its maximum in $B_R(0)$. We denote this point with x_0. We have there that

$$(i) \ \nabla g(x_0) = 0 \ \text{ and } \ (ii) \bar{L} g(x_0) \leq 0 \, .$$

We compute at the point x_0 and get from (i)

$$0 = D_i D_k u \, D_k u \, (u^2 + k) \, \eta^2 + |Du|^2 \, u \, D_i u \, \eta^2 + |Du|^2 \, (u^2 + k) \, \eta \, D_i \eta \, , \tag{3.6}$$

and from (ii)

$$\begin{aligned} 0 \geq \bar{a}^{ij} \Big(& D_{ij}(D_k u) D_k u (u^2 + k) \eta^2 + D_{ik} u D_{jk} u (u^2 + k) \eta^2 + 2 D_{ik} u D_k u D_j u \, u \eta^2 \\ & 2 D_{ik} u D_k u (u^2 + k) \eta D_j \eta + 2 D_{jk} u D_k u D_i u \, u \eta^2 + 2 |Du|^2 u D_{ij} u \, \eta^2 \\ & + |Du|^2 D_j u D_i u \, \eta^2 + 2 |Du|^2 u D_i u \eta D_j \eta + 2 D_k u D_{jk} u (u^2 + k) \eta D_i \eta \\ & + 2 |Du|^2 u D_j u \, \eta D_i \eta + |Du|^2 (u^2 + k)(D_j \eta D_i \eta + \eta D_{ij} \eta) \Big) \, . \end{aligned}$$

Using that $\bar{L}u = 0$ and the uniform ellipticity we see that this implies

$$\begin{aligned} 0 \geq & \lambda |D^2 u|^2 (u^2 + k) \eta^2 + \lambda |Du|^4 \eta^2 + \lambda |Du|^2 (u^2 + k) |D\eta|^2 \\ & + 4 \bar{a}^{ij} D_{ik} u D_k u D_j u u \eta^2 + 4 \bar{a}^{ij} D_{ik} u D_k u D_j \eta (u^2 + k) \eta \\ & + 4 \bar{a}^{ij} D_i u D_j \eta |Du|^2 \eta u + \bar{a}^{ij} D_{ij} \eta |Du|^2 (u^2 + k) \eta \end{aligned} \tag{3.7}$$

Using (3.6) we see that

$$\begin{aligned} 4 \bar{a}^{ij} D_{ik} u D_k u D_j \eta (u^2 + k) \eta &= -4 \bar{a}^{ij} D_i u D_j \eta |Du|^2 u \eta - 4 \bar{a}^{ij} D_i \eta D_j \eta |Du|^2 (u^2 + k) \\ &\geq -C |Du|^3 |u| |D\eta| \eta - C |Du|^2 (u^2 + k) |D\eta|^2 \, . \end{aligned} \tag{3.8}$$

By Young's inequality we can estimate

$$\begin{aligned} 4 \bar{a}^{ij} D_{ik} u D_k u D_j u \, u \eta^2 &\geq -C |D^2 u| |Du|^2 |u| \eta^2 \geq -\lambda k |D^2 u|^2 \eta^2 - \frac{C}{\lambda k} |Du^4| u^2 \eta^2 \\ &\geq -\lambda |D^2 u|^2 (u^2 + k) \eta^2 - \frac{\lambda}{2} |Du|^4 \eta^2 \, , \end{aligned} \tag{3.9}$$

if we choose $k = \frac{2C}{\lambda^2}\sup_{B_R(0)} u^2$. Using (3.8) and (3.9) in (3.7) we arrive at

$$\begin{aligned} 0 &\geq \frac{\lambda}{2}|Du|^4\eta^2 - C|Du|^3|u||D\eta|\eta - C|Du|^2(u^2+k)|D\eta|^2 \\ &\quad - C|Du|^2(u^2+k)\eta|D^2\eta| \\ &\geq \frac{\lambda}{4}|Du|^4\eta^2 - C|Du|^2(u^2+k)|D\eta|^2 \ - C|Du|^2(u^2+k)\eta|D^2\eta|\,, \end{aligned} \tag{3.10}$$

where we estimated

$$C|Du|^3|u||D\eta|\eta \leq \frac{\lambda}{4}|Du|^4\eta^2 + \frac{C}{\lambda}|Du|^2u^2|D\eta|^2 \ .$$

Note that $D_i\eta = D_i(\tilde{R}^2 - |x|^2) = -2x_i$ and $D_{ij}\eta = -2\delta_{ij}$ and thus

$$\eta \leq \tilde{R}^2\,, \qquad |D\eta|^2 \leq 4\tilde{R}^2\,, \qquad |D_{ij}\eta| \leq 2n\,.$$

Using this in (3.10) yields at the point x_0 (where we can w.l.o.g. assume that $|Du|(x_0) > 0$) that

$$\eta^2|Du|^2 \leq CR^2 \sup_{B_R(0)} u^2$$

and thus $\forall\, x \in B_R(0)$, since $k = \frac{2C}{\lambda^2}\sup_{B_R(0)} u^2$, that

$$g(x) \leq g(x_0) = |Du|^2(x_0)(u^2(x_0)+k)\eta^2(x_0) \leq CR^2(\sup_{B_R(0)} u^2)^2\,.$$

On the other hand we have for $0 < \theta < 1$ and $\forall\, x \in B_{\theta\tilde{R}}(0)$:

$$g(x) = |Du|^2(x)(u^2(x)+k)(\tilde{R}^2 - |x|^2)^2 \geq \frac{2C}{\lambda^2}|Du|^2(x) \sup_{B_R(0)} u^2(1-\theta^2)^2\tilde{R}^4\,.$$

Letting $\tilde{R} \nearrow R$ implies that $\forall\, x \in B_{\theta R}(0)$

$$|Du|^2(x) \leq \frac{C}{(1-\theta^2)^2}R^{-2} \sup_{B_R(x_0)} u^2\,.$$

Since $\bar{a}^{ij}$ is constant we can apply this result to $v = D_k u$ for $k = 1, \dots, n$ to see that

$$\sup_{B_{3R/4}(0)} |D^2u| \leq CR^{-1} \sup_{B_{4R/5}(0)} |Du| \leq CR^{-2} \sup_{B_R(0)} |u| \ .$$

Iterating this once more we arrive at the claimed estimate. □

Proof of Lemma 3.2.4: Recall that we want to show that for all $\varepsilon > 0$ and integers $1 \leq l \leq k$ there is a $C = C(\varepsilon, \mu, k, n)$, such that for all $h \in C^{k,\mu}(\bar{B}_R(x_0))$ it holds

$$R^l|D^lh|_{0;\,B_R(x_0)} \leq \varepsilon R^{k+\mu}[D^kh]_{\mu;\,B_R(x_0)} + C|h|_{0;\,B_R(x_0)}\,.$$

We will only prove here the case $l = 1, 2$, $k = 1, 2$, which are the cases needed in the previous proof. W.l.o.g. we can assume $x_0 = 0$.

$\underline{l = 1, k = 1:}$

Let $h \in C^{1,\beta}(\bar{B}_R(0))$, $\beta \in (0, 1]$, then for $1 \le j \le n$ we have

$$|D_j h(y) - D_j h(x)| \le |x - y|^\beta [D_j h]_{\beta;\, B_R(0)}\,.$$

Assume $|D_j h(x_1)| = \sup_{B_R(0)} |D_j h|$ and $x_1 \in \bar{B}_{\sigma/2}(y)$, where $B_{\sigma/2}(y) \subset B_R(0)$. Note that such a y exists $\forall\, \sigma \le 2R$. Thus we obtain $\forall\, w \in B_\sigma(y) \cap B_R(0)$ that

$$\big||D_j h(w)| - |D_j h(x_1)|\big| \le |D_j h(w) - D_j h(x_1)| \le \sigma^\beta [D_j h]_{\beta;\, B_R(0)}$$

which implies

$$|D_j h(x_1)| \le \inf_{B_{\sigma/2}(y)} |D_j h| + \sigma^\beta [D_j h]_{\beta;\, B_R(0)}\,.$$

This in turn gives that $\forall\, \sigma \in (0, R]\ \exists\, B_{\sigma/2}(y) \subset B_R(0)$ such that

$$|D_j h|_{0,B_R(0)} \le \inf_{B_{\sigma/2}(y)} |D_j h| + \sigma^\beta [D_j h]_{\beta;\, B_R(0)}\,.$$

We can assume that $\inf_{B_{\sigma/2}(y)} |D_j h| > 0$, otherwise we are already finished. We consider the line segment $\gamma = \{y + s e_j | s \in [-\sigma/4, \sigma/4]\} \subset B_{\sigma/2}(y)$ and compute

$$|h(y + \frac{\sigma}{4} e_j) - h(y - \frac{\sigma}{4} e_j)| = \Big| \int_\gamma D_j h\, dt \Big| \ge \frac{\sigma}{2} \inf_{B_{\sigma/2}(y)} |D_j h|\,,$$

which implies that $\inf_{B_{\sigma/2}(y)} |D_j h| \le 4\sigma^{-1} |h|_{0;\, B_R(0)}$. Putting this together we obtain

$$|D_j h|_{0;\, B_R(0)} \le \sigma^{-1} |h|_{0;\, B_R(0)} + \sigma^\beta [D_j h]_{\beta;\, B_R(0)} \qquad \forall\, \sigma \in (0, R]\,. \tag{3.11}$$

Choosing $\sigma = (\varepsilon/n)^{1/\beta} R$ for $0 < \varepsilon < 1$, and summing over j proves the statement for $l = 1, k = 1$.

Since $B_R(0)$ is convex we have that for $h \in C^2(\bar{B}_R(0))$ that $[Dh]_{1;\, B_R(0)} = |D^2 h|_{0;\, B_R(0)}$ and we obtain from (3.11) with $\beta = 1$ that

$$|Dh|_{0;\, B_R(0)} \le C\sigma^{-1} |h|_{0;\, B_R(0)} + \sigma |D^2 h|_{0;\, B_R(0)} \qquad \forall\, \sigma \in (0, R]\,. \tag{3.12}$$

$\underline{l = 1, k = 2:}$

We choose $\sigma = \varepsilon R$ for $0 < \varepsilon < 1$ in (3.12) to get

$$R|Dh|_{0;\, B_R(0)} \le C\varepsilon^{-1} |h|_{0;\, B_R(0)} + \varepsilon R^2 |D^2 h|_{0;\, B_R(0)} \qquad \forall\, \sigma \in (0, R]\,. \tag{3.13}$$

We now take (3.11), replacing h by $D_i h$, to get

$$|D^2 h|_{0;\, B_R(0)} \le 4n\sigma^{-1} |Dh|_{0;\, B_R(0)} + n\sigma^\beta [D^2 h]_{\beta;\, B_R(0)} \qquad \forall\, \sigma \in (0, R]\,. \tag{3.14}$$

Use this in (3.13) with $\sigma = \delta R$ to see

$$R|Dh|_{0;\, B_R(0)} \le C\varepsilon^{-1}|h|_{0;\, B_R(0)} + 4n\varepsilon\delta^{-1}R|Dh|_{0;\, B_R(0)} + n^\beta\varepsilon\delta^\beta R^2[D^2h]_{\beta;\, B_R(0)}\,.$$

For a given $0 < \varepsilon' < 1$ choose δ such that $\delta^\beta = \varepsilon'/(2n^\beta)$. Then choose ε such that $4n\varepsilon\delta^{-1} = 1/2$ and arrive at

$$\frac{1}{2}R|Dh|_{0;\, B_R(0)} \le C\varepsilon^{-1}|h|_{0;\, B_R(0)} + \frac{\varepsilon'}{2}R^2[D^2h]_{\beta;\, B_R(0)}\,.$$

<u>$l = 2, k = 2$:</u>

We use (3.13) in (3.14) to get the estimate

$$|D^2h|_{0;\, B_R(0)} \le 4n\sigma^{-1}\big(C\varepsilon^{-1}R^{-1}|h|_{0;\, B_R(0)} + \varepsilon R|D^2h|_{0;\, B_R(0)}\big) + n\sigma^\beta[D^2h]_{\beta;\, B_R(0)}$$

$\forall\, \sigma \in (0, R]$ and proceed as before. □

In the case that we have an equation in divergence form the following lemma is the essential estimate.

Lemma 3.2.5. *Let $u \in C^1(\mathbb{R}^n)$ be a weak solution of $D_i(\bar{a}^{ij}D_ju) = D_lf^l + g$, where the matrix $(\bar{a}^{ij})$ is constant, uniformly elliptic and satisfies $|\bar{a}^{ij}| < \Lambda$. If $[Du]_{\mu;\mathbb{R}^n} < \infty$ for some $0 < \mu < 1$, then*

$$[Du]_{\mu;\mathbb{R}^n} \le C\Big(\sum_{l=1}^n [f^l]_{\mu;\mathbb{R}^n} + [g]_{\mu;\mathbb{R}^n}\Big)$$

with a constant $C = C(n, \lambda, \Lambda, \mu)$.

Proof. Completely analogous to the proof of Lemma 3.2.3. □

Proof of Theorem 3.2.2. We write for simplicity $B_R = B_R(0)$. Recall that we assume that $u \in C^{2,\mu}(B_R)$ solves

$$\sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta}D^{\alpha+\beta}u = f\,,$$

with $f \in C^{0,\mu}(B_R)$. Further more we assume that

$$\text{(3.15)}\quad \sum_{|\alpha|,|\beta|\le 1} R^{2-|\alpha+\beta|}|a_{\alpha\beta}|_{0;\, B_R(0)} \le \Lambda_1 \;\text{ and }\; \sum_{|\alpha|,|\beta|\le 1} R^{2+\mu-|\alpha+\beta|}[a_{\alpha\beta}]_{0,\,\mu;\, B_R(0)} \le \Lambda_2\,,$$

with $\mu \in (0, 1)$. Trick: freeze the coefficients. To do so, let $\bar{a}_{\alpha\beta} := a_{\alpha\beta}(0)$, and write

$$\sum_{|\alpha|,|\beta|\le 1} \bar{a}_{\alpha\beta}D^{\alpha+\beta}u = \sum_{|\alpha|,|\beta|\le 1} (\bar{a}_{\alpha\beta} - a_{\alpha\beta})D^{\alpha+\beta}u + f\,,$$

Let us first assume that $R = 1$. We would like to apply Lemma 3.2.3: So choose $\varphi \in C_c^\infty(B_1)$, $0 \le \varphi \le 1$, $\varphi = 1$ on B_θ for $0 < \theta < 1$ and let $\tilde{u} = \varphi \cdot u \in C^{2,\mu}(\mathbb{R}^n)$:

$$\begin{aligned}
\sum_{|\alpha|=|\beta|=1} \bar{a}_{\alpha\beta} D^{\alpha+\beta}\tilde{u} = \varphi\Big(&\sum_{|\alpha|,|\beta|\le 1} (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) D^{\alpha+\beta} u + f\Big) \\
&- \varphi\Big(\sum_{|\alpha+\beta|\le 1} \bar{a}_{\alpha\beta} D^{\alpha+\beta} u\Big) \\
&+ \sum_{|\alpha|=|\beta|=1} \Big(\bar{a}_{\alpha\beta}\big(D^\alpha\varphi D^\beta u + D^\beta\varphi D^\alpha u + u D^{\alpha+\beta}\varphi\big)\Big) \\
= \varphi &\sum_{|\alpha|,|\beta|=1} (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) D^{\alpha+\beta} u - \varphi\Big(\sum_{|\alpha+\beta|\le 1} a_{\alpha\beta} D^{\alpha+\beta} u\Big) + \varphi f \\
&+ \sum_{|\alpha|=|\beta|=1} \Big(\bar{a}_{\alpha\beta}\big(D^\alpha\varphi D^\beta u + D^\beta\varphi D^\alpha u + u D^{\alpha+\beta}\varphi\big)\Big) =: g(x)
\end{aligned}$$

Recall that for a set A we have

$$[f+g]_{\mu;A} \le [f]_{\mu;A} + [g]_{\mu;A} \quad \text{and} \quad [f\cdot g]_{\mu;A} \le |f|_{0;A}\cdot[g]_{\mu;A} + [f]_{\mu;A}\cdot|g|_{0;A}\,,$$

and thus

$$\begin{aligned}
[g]_{\mu;\mathbb{R}^n} = [g]_{\mu;B_1} \le\ & |(\bar{a}^{ij} - a^{ij})\varphi|_{0;B_1}[D^2u]_{\mu;B_1} + [(\bar{a}^{ij} - a^{ij})\varphi]_{\mu;B_1}|D^2u|_{0;B_1} \\
&+ C|u|_{1;B_1}[\varphi\, a_{\alpha\beta}]_{\mu;B_1} + C|u|_{1,\mu;B_1}|\varphi\, a_{\alpha\beta}|_{0;B_1} \\
&+ |f|_{0;B_r}[\varphi]_{\mu;B_1} + [f]_{0,\mu;B_1}|\varphi|_{0;B_1} + C|\bar{a}_{\alpha\beta}D\varphi|_{0;B_1}[Du]_{\mu;B_1} \\
&+ C[\bar{a}_{\alpha\beta}D\varphi]_{\mu;B_1}|Du|_{0;B_1} + C|\bar{a}_{\alpha\beta}D^2\varphi|_{0;B_1}[u]_{\mu;B_1} \\
&+ C[\bar{a}_{\alpha\beta}D^2\varphi]_{\mu;B_1}|u|_{0;B_1}\,,
\end{aligned}$$

which gives

$$\begin{aligned}
[g]_{\mu;\mathbb{R}^n} = [g]_{\mu;B_1} &\le |(\bar{a}^{ij} - a^{ij})\varphi|_{0;B_1}[D^2u]_{\mu;B_1} + C|u|_{2;B_1} + C|f|_{0,\mu;B_1} \\
&\le \big(|(\bar{a}^{ij} - a^{ij})|_{0;B_1} + \varepsilon\big)[D^2u]_{\mu;B_1} + C|u|_{0;B_1} + C_\varepsilon|f|_{0,\mu;B_1}\,.
\end{aligned}$$

where we used the interpolation lemma, Lemma 3.2.4, in the last step. Note that the constant C in the above estimate just depends on n, Λ_1, Λ_2 and φ. Applying Lemma 3.2.3 we obtain

$$[D^2u]_{\mu;B_\theta} \le C\big(|(\bar{a}^{ij} - a^{ij})|_{0;B_1} + \varepsilon\big)[D^2u]_{\mu;B_1} + C_\varepsilon|u|_{0;B_1} + C|f|_{0,\mu;B_1} \tag{3.16}$$

Let us now assume that R is again arbitrary. So for $u : B_R \to \mathbb{R}$ define on B_1:

$$\hat{u}(x) := R^{-2}u(R\cdot x)\,, \qquad \hat{f}(x) = f(R\cdot x)\,,$$

and

$$\hat{a}_{\alpha\beta}(x) = R^{2-|\alpha+\beta|}a_{\alpha\beta}(R\cdot x)$$

It is easily checked that

$$\sum_{|\alpha|,|\beta|\le 1} \hat{a}_{\alpha\beta} D^{\alpha+\beta}\hat{u} = \hat{f}$$

on B_1. Note that the assumption (3.15) implies that $\hat{a}_{\alpha\beta}$ satisfies (3.15) with $R = 1$. Furthermore we have

$$\begin{aligned} |\hat{u}|_{0;\,B_1} &= R^{-2}|u|_{0;\,B_R} \qquad [D^2\hat{u}]_{\mu;\,B_\theta} = R^\mu [D^2 u]_{\mu;\,B_{\theta R}} \\ |\hat{f}|_{0;\,B_1} &= |f|_{0;\,B_R} \qquad [\hat{f}]_{\mu;\,B_1} = R^\mu [f]_{\mu;\,B_R}\,, \end{aligned}$$

and so we obtain from (3.16)

$$\begin{aligned} R^\mu [D^2 u]_{\mu;\,B_{\theta R}} \le\ & C\big(|(\bar{a}^{ij} - a^{ij})|_{0;\,B_R} + \varepsilon\big) R^\mu [D^2 u]_{\mu;\,B_R} \\ & + C_\varepsilon R^{-2}|u|_{0;\,B_R} + C|f|_{0;\,B_R} + CR^\mu [f]_{\mu;\,B_R} \end{aligned} \tag{3.17}$$

Difficulty: the left hand side is only a norm on $B_{\theta R}$, but we want B_R, and $|(\bar{a}^{ij} - a^{ij})|_{0;\,B_R}$ is only small if R is small. So we will need a covering argument.

So let $\delta \in (0, 1)$ and note that on any $B_{\delta R}(x) \subset B_R$ we have

$$|(\bar{a}^{ij} - a^{ij})|_{0;\,B_{\delta R}(x)} \le \Lambda_2 \delta^\mu\,.$$

This implies for any $B_\rho(y) \subset B_R$ with $\rho \le \delta R$:

$$\begin{aligned} [D^2 u]_{\mu;\,B_{\theta\rho}(y)} \le\ & C(\varepsilon + \Lambda_2\delta^\mu)[D^2 u]_{\mu;\,B_\rho(y)} + C_\varepsilon \rho^{-2-\mu}|u|_{0;\,B_\rho(y)} \\ & + C\rho^{-\mu}|f|_{0;\,B_\rho(y)} + C[f]_{\mu;\,B_\rho(y)} \end{aligned}$$

Now define $S : \{A \subset B_R \,|\, A \text{ convex}\} \to \mathbb{R}^+$ by $S(A) := [D^2 u]_{\mu;\,A}$. Note that S is monotone and subadditive. It satisfies $\forall\, \rho \in (0, \delta R]$ and y such that $B_\rho(y) \subset B_R$:

$$\rho^{2+\mu} S(B_{\theta\rho}(y)) \le C(\varepsilon + \Lambda_2\delta^\mu)\rho^{2+\mu} S(B_\rho(y)) + \gamma\,, \tag{3.18}$$

where $\gamma = C_\varepsilon |u|_{0;\,B_R} + C\big(R^2|f|_{0;\,B_R} + R^{2+\mu}[f]_{\mu;\,B_R(0)}\big)$. We want to apply the following covering lemma:

Lemma 3.2.6. *Let $S : \{A \subset B_R \,|\, A \text{ convex}\} \to \mathbb{R}^+$ be a monotone subadditive function. Then there exists for given $\theta \in (0, 1)$, $\delta \in (0, 1]$, $\gamma \ge 0, l \ge 1$ an $\varepsilon_0 = \varepsilon_0(n, \theta, l) > 0$ such that if*

$$\rho^l S(B_{\theta\rho}(y)) \le \varepsilon_0 \rho^l S(B_\rho(y)) + \gamma \qquad \forall\, B_\rho(y) \subset B_R,\ \rho \le \delta R\,,$$

then

$$R^l S(B_{\theta R}) \le C\gamma\,,$$

where $C = C(n, \theta, l, \delta)$.

This implies the interior Hölder estimate for D^2u: In (3.18) choose $l = 2+\mu$ and $\varepsilon, \delta > 0$ in such a way that $C(\varepsilon + \Lambda_2\delta^\mu) < \varepsilon_0$. This yields

$$[D^2u]_{\mu;\,B_{\theta R}} \leq C\big(R^{-2-\mu}|u|_{0,\,B_R} + R^{-\mu}|f|_{0;\,B_R} + [f]_{\mu;\,B_R(0)}\big)\,,$$

where $C = C(n, \theta, \mu, \lambda, \Lambda_1, \Lambda_2)$. □

Proof of Lemma 3.2.6. We set

$$Q := \sup_{\substack{B_{\theta\rho}(y)\subset B_R\\ \theta\rho\leq\delta R}} \rho^l S(B_{\theta\rho(y)})\,.$$

This implies

$$(\theta\rho)^l \cdot S(B_{\theta^2\rho}(y)) \leq \varepsilon_0\theta^l Q + \gamma \tag{3.19}$$

for all $B_{\theta\rho}(y) \subset B_R$ and $\theta\rho \leq \delta R$. Now let $B_\rho(y) \subset B_R$ be arbitrary with $\rho \leq \delta R$. Note that we can find $N \leq c(\theta, n)$ and points y_i, $i = 1, \ldots, N$, $y_i \in B_{\theta\rho}(y)$ such that

$$B_{\theta\rho}(y) \subset \cup_i B_{\theta^2(1-\theta)\rho}(y_i)\,.$$

Then apply (3.19) to $B_{(1-\theta)\rho}(y_i)$ to get

$$(\theta(1-\theta)\rho)^l \cdot S\big(B_{\theta^2(1-\theta)\rho}(y_i)\big) \leq \varepsilon_0\theta^l Q + \gamma$$

which is equivalent to

$$\rho^l \cdot S\big(B_{\theta^2(1-\theta)\rho}(y_i)\big) \leq (1-\theta)^{-l}\varepsilon_0 Q + (\theta(1-\theta))^{-l}\gamma\,.$$

Since S is subadditive this implies

$$\rho^l \cdot S\big(B_{\theta\rho}(y)\big) \leq c(\theta, n)(1-\theta)^{-l}\varepsilon_0 Q + c(\theta, n)(\theta(1-\theta))^{-l}\gamma\,,$$

which in turn gives

$$Q \leq c(\theta, n)(1-\theta)^{-l}\varepsilon_0 Q + c(\theta, n)(\theta(1-\theta))^{-l}\gamma\,.$$

Now choose $\varepsilon_0 = (1-\theta)^l/(2c(\theta, n))$ to arrive at

$$Q \leq 2c(\theta, n)(\theta(1-\theta))^{-l} \cdot \gamma = C(\theta, n, l)\gamma$$

and thus

$$\rho^l S(B_{\theta\rho}(y)) \leq C(\theta, n, l)\gamma \ \ \forall\, B_\rho(y) \subset B_R,\ \rho \leq \delta R\ .$$

Finally cover $B_{\theta R}$ by finitely many $B_{\theta\rho}(y)$, $\rho = \delta R$, where the number of balls needed only depends on δ and n, to arrive at

$$R^l S(B_{\theta R}) \leq \tilde{C}\gamma\,,$$

with $\tilde{C} = \tilde{C}(\theta, n, l, \delta)$. □

In the case that the equation is in divergence form we will sketch the first part of the proof:

Proof of Theorem 3.2.1. We assume that $u \in C^{1,\mu}(\bar{B}_R)$, $B_R := B_R(0)$, is a solution to $(\star\star)$. As in the proof of Thorem 3.2.2 we let $\bar{a}_{\alpha\beta} := a_{\alpha\beta}(0)$ and we rewrite $(\star\star)$ as

$$\begin{aligned}\sum_{|\alpha|,|\beta|=1} D^\beta(\bar{a}_{\alpha\beta}D^\alpha u) = &\sum_{|\alpha|,|\beta|=1} D^\beta\big((\bar{a}_{\alpha\beta}-a_{\alpha\beta})D^\alpha u\big) - \sum_{|\alpha+\beta|\le 1} D^\beta(a_{\alpha\beta}D^\alpha u)\\ &+\sum_{|\beta|\le 1} D^\beta f_\beta \,.\end{aligned} \tag{3.20}$$

where we assume that

$$\sum_{|\alpha|,|\beta|\le 1} R^{2-|\alpha+\beta|}|a_{\alpha\beta}|_{0;\,B_R(0)} \le \Lambda_1 \quad \text{and} \quad \sum_{|\alpha|,|\beta|\le 1} R^{2+\mu-|\alpha+\beta|}[a_{\alpha\beta}]_{0,\,\mu;\,B_R(0)} \le \Lambda_2\,,$$

with $\mu \in (0,1)$. Again, as before, we assume first that $R = 1$ and choose $\varphi \in C_c^\infty(B_1)$, $0 \le \varphi \le 1$, $\varphi = 1$ on B_θ for $0 < \theta < 1$ and let $\tilde{u} = \varphi \cdot u \in C^{1,\mu}(\mathbb{R}^n)$. Multiplying (3.20) with φ we obtain

$$\begin{aligned}\sum_{|\alpha|,|\beta|=1} D^\beta(\bar{a}_{\alpha\beta}D^\alpha \tilde{u}) = &\sum_{|\alpha|,|\beta|=1} D^\beta\big(\varphi(\bar{a}_{\alpha\beta}-a_{\alpha\beta})D^\alpha u\big) + \sum_{|\alpha|,|\beta|=1} D^\beta(\bar{a}_{\alpha\beta}uD^\beta u)\\ &-\sum_{|\alpha+\beta|\le 1} D^\beta(a_{\alpha\beta}\varphi D^\alpha u) + \sum_{|\beta|\le 1} D^\beta(\varphi f_\beta)\\ &-\sum_{|\alpha|,|\beta|=1} \bar{a}_{\alpha\beta}D^\beta\varphi D^\alpha u + \sum_{|\alpha+\beta|\le 1} a_{\alpha\beta}D^\beta\varphi D^\alpha u\\ &-\sum_{|\beta|\le 1} f_\beta D^\beta\varphi =: \sum_{|\beta|\le 1} D^\beta g_\beta\,,\end{aligned} \tag{3.21}$$

where $g_\beta = g_\beta(u,f)$. From Lemma 3.2.5 we obtain the estimate

$$[u]_{1,\,\mu;\,B_\theta} \le C\sum_{|\beta|\le 1}[g_\beta]_{\mu;\,\mathbb{R}^n} = C\sum_{|\beta|\le 1}[g_\beta]_{\mu;\,B_1}\,,$$

and we can proceed as in the proof of Theorem 3.2.2. □

3.3 Boundary and global Schauder estimates

Definition: Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that $\Omega \cap B_R(x_0)$, $x_0 \in \partial\Omega$ has $C^{k,\mu}$-boundary, if there is an orthogonal transformation Q, s.t. for $y = Q(x-x_0)$ and a function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$, $\psi \in C^{k,\mu}(\mathbb{R}^{n-1})$ it holds

$$\Omega \cap B_R(x_0) = \{x_0 + Q^{-1}(y)\,|\, y_n > \Psi(y_1,\ldots,y_{n-1})\} \cap B_R(x_0)\,,$$

and

$$\partial\Omega \cap B_R(x_0) = \{x_0 + Q^{-1}(y) \,|\, y_n = \Psi(y_1, \dots, y_{n-1})\} \cap B_R(x_0)\,.$$

One says then that the map $\xi : x \mapsto \xi(x) = \big(y_1, \dots, y_{n-1}, y_n - \Psi(y_1, \dots, y_{n-1})\big)$ 'flattens out' the boundary near x_0. Furthermore, we say that $\partial\Omega$ is of the class $C^{k,\mu}$ if there is an $R > 0$ such that the above holds for every $B_R(x_0)$, $x_0 \in \partial\Omega$ with Ψ_{x_0} uniformly bounded in $C^{k,\mu}(\mathbb{R}^{n-1})$, independent of x_0.

Theorem 3.3.1 (local Schauder estimates at the boundary)**.** *Let $x_0 \in \partial\Omega$ and $\partial\Omega \cap B_R(x_0)$ be of class $C^{2,\mu}$ and $u \in C^{2,\mu}(\bar\Omega \cap B_R(x_0))$ a solution of*

$$Lu = \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta} D^{\alpha+\beta} u = f\,, \tag{3.22}$$

where $|a_{\alpha\beta}|_{0,\,\mu;\,\Omega\cap B_R(x_0)} \le \Lambda$, $a^{ij} \ge \lambda \cdot \delta^{ij}$, $|f|_{0,\,\mu;\,\Omega\cap B_R(x_0)} < \infty$ and $u|_{\partial\Omega\cap B_R(x_0)} = 0$. Then there is a $\theta_0 \in (0,1]$ such that $\forall\, \theta \in (0,\theta_0)$ there is a constant $C > 0$ depending only on $n, \lambda, \Lambda, \mu, R, \theta, \partial\Omega \cap B_R(x_0)$ s.t.

$$|u|_{2,\,\mu;\,\Omega\cap B_{\theta R}(x_0)} \le C(|u|_{0;\,\Omega\cap B_R(x_0)} + |f|_{0,\,\mu;\,\Omega\cap B_R(x_0)})\,.$$

Remark 3.3.2: *i*) If $\varphi \in C^{2,\mu}(\bar\Omega \cap B_R(x_0))$ and $u|_{\partial\Omega} = \varphi$, then $L(u-\varphi) = f - L\varphi$, so $(u-\varphi)|_{\partial\Omega} = 0$ and

$$|u|_{2,\,\mu;\,\Omega\cap B_{\theta R}(x_0)} \le C(|u|_{0;\,\Omega\cap B_R(x_0)} + |f|_{0,\,\mu;\,\Omega\cap B_R(x_0)} + |\varphi|_{2,\,\mu;\,\Omega\cap B_R(x_0)})\,.$$

ii) If $\partial\Omega$ is of class $C^{2,\mu}$ and $\varphi \in C^{2,\mu}(\partial\Omega)$, then there is an extension $\tilde\varphi \in C^{2,\mu}(\bar\Omega)$, s.t.

$$|\tilde\varphi|_{2,\,\mu;\,\Omega} \le C(\Omega, n, \mu)|\phi|_{2,\,\mu;\,\partial\Omega}\,.$$

iii) The constant θ_0 depends on 'how close' $\partial\Omega \cap B_R(x_0)$ is to a hyperplane. So if $R > 0$ is sufficiently small, depending on $\partial\Omega$, then θ_0 can be chosen arbitrary close to 1.

Proof. We argue analogously to the proof of Theorem 3.2.2. We first need to show that for solutions $u \in C^2(\overline{\mathbb{R}^n_+})$, of

$$\bar a^{ij} D_{ij} u = f\,,$$

where $(\bar a^{ij})$ is constant, $u|_{\{x_n=0\}} = 0$, $\mathbb{R}^n_+ = \{x_n > 0\}$, $[D^2u]_{\mu;\,\mathbb{R}^n_+} < \infty$ it holds

$$[D^2u]_{\mu;\,\mathbb{R}^n_+} \le C[f]_{\mu;\,\mathbb{R}^n_+}\,, \tag{3.23}$$

for some $C = C(n, \mu, \lambda, \Lambda)$. This is done exactly as in the proof of Lemma 3.2.3. The only tricky point is that we need interior gradient estimates up to the boundary, that is for some $\gamma \in (0,1)$ we want to show

$$\sup_{B_{\gamma R}(x_0)\cap\mathbb{R}^n_+} |D^m u| \le CR^{-m} \sup_{B_R(x_0)\cap\mathbb{R}^n_+} |u| \tag{3.24}$$

for all $B_R(x_0) \subset \mathbb{R}^n$, provided $u \in C^m(\overline{\mathbb{R}^n_+})$ solves

$$\bar{a}^{ij} D_{ij} u = 0\,,$$

with $u = 0$ on $\partial\mathbb{R}^n_+$. This is proven as follows: Since $\bar{a}^{ij}$ is symmetric and positive definite, there is an orthogonal transformation Q of $\mathbb{R}^n$ such that

$$Q(\bar{a}^{ij})Q^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

with $\lambda_i > 0$, $i = 1, \ldots, n$. Then let $P = \mathrm{diag}((\lambda_1)^{-1/2}, \ldots, (\lambda_n)^{-1/2})$, which gives

$$PQ(\bar{a}_{ij})(PQ)^T = PQ(\bar{a}_{ij})Q^{-1}P = (\delta_{ij})\,.$$

We define $\tilde{u}$ on $Q\mathbb{R}^n_+$ by $u(x) = \tilde{u}(PQ(x))$, which gives

$$(D_{ij}u) = \Big(\sum_{l,k}(PQ)_{li}(PQ)_{kj}D_{lk}\tilde{u}\Big) = (PQ)^T D^2\tilde{u} PQ\,,$$

and

$$\bar{a}^{ij}D_{ij}u = \mathrm{tr}\big(PQ(\bar{a}_{ij})(PQ)^T D^2\tilde{u}\big) = \Delta\tilde{u} = 0\,.$$

In addition $\tilde{u}$ is again defined on a half-space H, with $\tilde{u}|_{\partial H} = 0$. By elliptic regularity we have $u \in C^\infty(\bar{H})$. Extending $\tilde{u}$ by odd reflection at ∂H onto all $\mathbb{R}^n$ we can apply the interior estimates for harmonic functions (or claim (3.5)), to obtain the estimate (3.24) for $\tilde{u}$ and $\gamma = 1/2$. Transforming back to u we the get claimed estimate (3.24) for u.

After then proving (3.23) we proceed as follows: On $B_R(x_0)\cap\Omega$, flatten out the boundary by defining

$$\hat{u}(\xi(x)) = u(x) \quad \text{and} \quad \hat{f}(\xi(x)) = f(x)\ .$$

It is then easily checked that $\hat{u}$ satisfies again an equation as in (3.22), where $\hat{\lambda} \geq c^{-1}\lambda$, $\hat{\Lambda} \leq C\Lambda$ for some $c = c(\xi) > 0$. Again as in the proof of Theorem 3.2.2 we assume first that $R = 1$ and take $\varphi \in C^\infty_c(B_1)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ on B_θ for $0 < \theta < 1$ and let $\tilde{u} = \varphi \cdot \hat{u}$. Note that x_0 is mapped to 0 under ξ. Arguing as in the proof there, using (3.23), and scaling back to arbitrary R, we obtain the estimate

$$\begin{aligned}\rho^\mu[D^2\hat{u}]_{\mu;\,B_{\theta\rho}(y)\cap\mathbb{R}^n_+} \leq\ & C\big(|(\bar{a}^{ij} - a^{ij})|_{0;\,B_\rho(y)\cap\mathbb{R}^n_+} + \varepsilon\big)\rho^\mu[D^2\hat{u}]_{\mu;\,B_\rho(y)\cap\mathbb{R}^n_+}\\ &+ C\rho^{-2}|\hat{u}|_{0;\,B_\rho(y)\cap\mathbb{R}^n_+} + C|\hat{f}|_{0;\,B_\rho(y)\cap\mathbb{R}^n_+} + C\rho^\mu[\hat{f}]_{\mu;\,B_\rho(y)\cap\mathbb{R}^n_+}\,,\end{aligned}$$

for any $B_\rho(y) \subset B_R$. Using a covering argument as in the proof there, we obtain

$$|\hat{u}|_{2,\,\mu;\,B_{\theta R}\cap\mathbb{R}^n_+} \leq C(|\hat{u}|_{0;\,B_R\cap\mathbb{R}^n_+} + |\hat{f}|_{0,\,\mu;\,B_R\cap\mathbb{R}^n_+})\,.$$

Then finally note that there is a constant $C = C(\xi) > 0$ and $\gamma = \gamma(\xi) \in (0,1)$ such that

$$C^{-1}|u|_{2,\,\mu;\,B_{\gamma\theta R}(x_0)\cap\Omega} \leq |\hat{u}|_{2,\,\mu;\,B_{\theta R}\cap\mathbb{R}^n_+} \leq C|u|_{2,\,\mu;\,B_{\gamma^{-1}\theta R}(x_0)\cap\Omega}$$

and similarly for all other norms. Adjusting θ and R accordingly, this implies the claimed estimate in the theorem. □

Theorem 3.3.3 (Global Schauder estimates)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $\partial\Omega$ of class $C^{2,\mu}$ and $u \in C^{2,\mu}(\bar{\Omega})$ be a solution to $Lu = f$ as in (3.22), where $f \in C^{\mu}(\bar{\Omega})$, $|a_{\alpha\beta}|_{\mu;\,\Omega} \leq \Lambda$, and $a^{ij} \geq \lambda \cdot \delta^{ij}$ for some $\lambda > 0$. Let $\varphi \in C^{2,\mu}(\bar{\Omega})$ and assume $u = \varphi$ on $\partial\Omega$. Then*

$$|u|_{2,\,\mu;\,\Omega} \leq C(|u|_{0;\,\Omega} + |\varphi|_{2,\,\mu;\,\Omega} + |f|_{0,\,\mu;\,\Omega})\,, \tag{3.25}$$

where $C = C(n, \mu, \lambda, \Lambda, \Omega)$.

Proof. We can first reduce to the case that $u = 0$ on $\partial\Omega$, i.e. $\varphi \equiv 0$. To do so, set $\tilde{u} = u - \varphi$ and compute

$$L\tilde{u} = Lu - L\varphi = f - L\varphi = \tilde{f}\,.$$

Then apply (3.25) with $\varphi \equiv 0$ to see

$$|\tilde{u}|_{2,\,\mu;\,\Omega} \leq C(|\tilde{u}|_{0;\,\Omega} + |\tilde{f}|_{0,\,\mu;\,\Omega})\,,$$

which yields

$$\begin{aligned} |u|_{2,\,\mu;\,\Omega} \leq |\tilde{u}|_{2,\,\mu;\,\Omega} + |\varphi|_{2,\,\mu;\,\Omega} &\leq C(|\tilde{u}|_{0;\,\Omega} + |\tilde{f}|_{0,\,\mu;\,\Omega}) + |\varphi|_{2,\,\mu;\,\Omega} \\ &\leq C(|u|_{0;\,\Omega} + |\varphi|_{2,\,\mu;\,\Omega} + |f|_{0,\,\mu;\,\Omega})\,. \end{aligned}$$

Thus we can in the following assume that $\varphi \equiv 0$. Since $\partial\Omega$ is of class $C^{2,\mu}$ there exists a $R_0 > 0$ such that the conclusions of Theorem 3.3.1 hold with $\theta = 3/4$ for every $x_0 \in \partial\Omega$. Then set

$$\Omega_{R_0} := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) \geq R_0/2\}\,.$$

We claim first that

$$|Du|(x) + |D^2u|(x) \leq C(|u|_{0;\,\Omega} + |f|_{0,\,\mu;\,\Omega}) \tag{3.26}$$

for all $x \in \Omega$. We distinguish two cases:

i) $\operatorname{dist}(x, \partial\Omega) \geq R_0/2$. Then $B_{R_0/2}(x) \subset \Omega$ and by Theorem 3.2.2 the claim holds with $C = C(n, 3/4, \lambda, \Lambda, \mu, R_0/2)$.

ii) $\operatorname{dist}(x, \partial\Omega) < R_0/2$. Then there is a point $x_0 \in \partial\Omega$ s.t. $x \in B_{\frac{3}{4}R_0}(x_0) \cap \Omega$. Thus by Theorem 3.3.1 again the claim holds with $C = C(n, 3/4, \lambda, \Lambda, \mu, R_0)$.

Now consider $x, y \in \Omega$, $x \neq y$. We distinguish three cases:

i) $x, y \in \Omega_{R_0}$.

ii) $\operatorname{dist}(x, y) \geq R_0/4$ and x or $y \notin \Omega_{R_0}$.

iii) $\operatorname{dist}(x, y) < R_0/4$ and x or $y \notin \Omega_{R_0}$.

Case i): If $\mathrm{dist}(x,y) < R_0/4$, then $B_{R_0/2}(x) \subset \Omega$ and $y \in B_{R_0/4}(x)$ and Theorem 3.2.2 gives

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\mu} \le [D^2u]_{\mu;\, B_{R_0/4}(x)} \le C(|u|_{0;\, B_{R_0/2}(x)} + |f|_{0,\,\mu;\, B_{R_0/2}(x)}) \le C(|u|_{0;\,\Omega} + |f|_{0,\,\mu;\,\Omega}) \ .$$

If $\mathrm{dist}(x,y) \ge R_0/4$ we have

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\mu} \le 2(R_0/4)^{-\mu} \sup_\Omega |D^2u| \le C(|u|_{0;\,\Omega} + |f|_{0,\,\mu;\,\Omega}) \ .$$

Case ii): As the previous step.

Case iii): Observe that there is a $x_0 \in \partial\Omega$ s.t. $x, y \in B_{\frac{3}{4}R_0}(x_0)$. Apply Theorem 3.3.1.

This implies

$$[D^2u]_{\mu;\,\Omega} \le C(|u|_{0;\,\Omega} + |f|_{0,\,\mu;\,\Omega}) \,,$$

which implies together with (3.26) that

$$|u|_{2,\,\mu;\,\Omega} \le C(|u|_{0;\,\Omega} + |f|_{0,\,\mu;\,\Omega}) \,.$$

□

To establish an existence theorem we need to approximate the domain Ω with $\partial\Omega$ of class $C^{2,\mu}$ by domains Ω_k with $\partial\Omega_k$ of class C^∞.

Definiton: We say that a sequence of domains Ω_k with $\partial\Omega_k$ of class $C^{2,\mu}$ converges to a domain Ω with $\partial\Omega$ of class $C^{2,\mu}$ in the $C^{2,\mu'}$-sense ($0 < \mu' \le \mu$), if

i) $\exists\, \varepsilon_k \searrow 0$ s.t.

$$\{x \in \Omega \mid \ \mathrm{dist}(x, \partial\Omega) > \varepsilon_k\} \subset \Omega_k \subset \{x \in \mathbb{R}^n \mid \ \mathrm{dist}(x, \Omega) < \varepsilon_k\} \,.$$

i.e. they converge in Hausdorff-distance.

ii) $\exists\, R_0 > 0$ s.t. $\forall\, x_0 \in \partial\Omega$ there is a coordinate system s.t.

$$\Omega \cap B_{R_0}(x_0) = \{y \in B_{R_0}(x_0) \,|\, y_n > \Psi_{x_0}(y_1, \dots, y_{n-1}\} \text{ with } \Psi_{x_0} \in C^{2,\mu}(\mathbb{R}^{n-1}) \ ,$$

$$\Omega_k \cap B_{R_0}(x_0) = \{y \in B_{R_0}(x_0) \,|\, y_n > \Psi_{x_0,k}(y_1, \dots, y_{n-1}\} \text{ with } \Psi_{x_0,k} \in C^{2,\mu}(\mathbb{R}^{n-1}) \ ,$$

where $|\Psi_{x_0,k}|_{2,\,\mu;\,\mathbb{R}^{n-1}}$ is bounded independently of k and x_0, and $|\Psi_{x_0,k} - \Psi_{x_0}|_{2,\,\mu';\,\mathbb{R}^{n-1}} \to 0$, uniformly in x_0.

Remark 3.3.4: i) If we have the situation as above, the constant in the global Schauder estimates can be chosen independently of Ω_k.

ii) If $\Omega \subset \mathbb{R}^n$ is a domain with $\partial\Omega$ of class $C^{2,\mu}$, then there exist $\Omega_k \subset \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > 1/k\}$ for $k \ge k_0$ such that the Ω_k converge to Ω in the $C^{2,\mu'}$-sense for some $0 < \mu' < \mu$, and $\partial\Omega_k$ is of class C^∞ for all k. See exercises.

Theorem 3.3.5 (Existence of solutions in $C^{2,\mu}(\bar{\Omega})$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ of class $C^{2,\mu}$, $f \in C^{0,\mu}(\bar{\Omega}), \varphi \in C^{2,\mu}(\bar{\Omega})$ and*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu \tag{3.27}$$

with $c \le 0$ and $|a^{ij}, b^i, c|_{0,\mu;\Omega} \le \Lambda$, $a^{ij} \ge \lambda \cdot \delta^{ij}$ for some $\lambda > 0$. Then there exists a unique solution $u \in C^{2,\mu}(\bar{\Omega})$ of

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

and u satisfies

$$|u|_{2,\mu;\Omega} \le C(|u|_{0;\Omega} + |f|_{0,\mu;\Omega} + |\varphi|_{2,\mu;\Omega}) \le C(|f|_{0,\mu;\Omega} + |\varphi|_{2,\mu;\Omega})$$

where $C = C(n, \mu, \lambda, \Lambda, \Omega)$, and we applied Theorem 3.1.4 in the last estimate.

Proof. As before by considering $\tilde{u} = u - \varphi$ and $\tilde{f} = f - L\varphi$ we can reduce to the case $\varphi \equiv 0$. Choose a sequence of sets as in Remark 3.3.4 *ii*). Let ϕ_ε be the standard mollifier with $\operatorname{supp}(\phi_\varepsilon) \subset B_\varepsilon(0)$. Then we set

$$a_k^{ij} := \phi_{1/k} * a^{ij}, \quad b_k^i := \phi_{1/k} * b^i, \quad c_k := \phi_{1/k} * c, \quad f_k := \phi_{1/k} * f.$$

Then $a_k^{ij}, b_k^i, c_k, f_k \in C^\infty(\bar{\Omega}_k)$, $a_k^{ij} \ge \lambda \cdot \delta^{ij}$ and $c_k \le 0$. Furthermore

$$a_k^{ij} \to a^{ij}, \quad b_k^i \to b^i, \quad c_k \to c, \quad f_k \to f$$

in $C^0_{\text{loc}}(\Omega)$ and $|a_k^{ij}, b_k^i, c_k|_{0,\mu;\Omega_k} \le \Lambda$, $|f_k|_{0,\mu;\Omega_k} \le |f|_{0,\mu;\Omega}$ for all $k \ge k_0$. By PDE I (Fredholm alternative, strong maximum principle for weak solutions, and regularity theory) there exists a unique solution $u_k \in C^\infty(\bar{\Omega}_k)$ of

$$L_k u_k = f_k \quad \text{on } \Omega_k$$

with $u_k = 0$ on $\partial\Omega_k$. By Theorem 3.3.3 and Theorem 3.1.4 we have the estimates

$$|u_k|_{2,\mu;\Omega_k} \le C|f_k|_{0,\mu;\Omega_k} \le C|f|_{0,\mu;\Omega},$$

where the constant C does not depend on k. So we can apply Arzela-Ascoli to extract a subsequence converging in C^2_{loc} to a solution $u \in C^{2,\mu}(\bar{\Omega})$. □

Theorem 3.3.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ of class $C^{2,\mu}$. Let Lu be as in* (3.27) *with $c \le 0$ and the bounds on the coefficients as stated there. Then the operator*

$$L\colon C^{2,\mu}(\bar{\Omega}) \to C^{0,\mu}(\bar{\Omega} \times C^{2,\mu}(\partial\Omega)), u \mapsto (Lu, u|_{\partial\Omega})$$

is an isomorphism of Banach spaces.

Proof. Obviously L is linear and continuous:

$$\|(Lu, u|_{\partial\Omega})\|_{C^{0,\mu}(\bar{\Omega})\times C^{2,\mu}(\partial\Omega)} \leq C \cdot \|u\|_{C^{2,\mu}(\bar{\Omega})}$$

and there is an inverse mapping

$$L^{-1} : C^{0,\mu}(\bar{\Omega}) \times C^{2,\mu}(\partial\Omega) \to C^{2,\mu}(\bar{\Omega}) \,,$$

given as follows: Let $f \in C^{0,\mu}(\bar{\Omega}), \varphi \in C^{2,\mu}(\partial\Omega)$. Extend φ to $\tilde{\varphi} \in C^{2,\mu}(\bar{\Omega})$ with $\|\tilde{\varphi}\|_{C^{2,\mu}(\bar{\Omega})} \leq C \,\|\varphi\|_{C^{2,\mu}(\partial\Omega)}$. Note that C does not depend on φ and this operation of extension can be made to be linear in φ. By Theorem 3.3.5 there is a unique solution $u \in C^{2,\mu}(\bar{\Omega})$ of $Lu = f$ on $\bar{\Omega}$ with $u = \tilde{\varphi} = \varphi$ on $\partial\Omega$. Then define

$$L^{-1}(f, \varphi) := u \,.$$

This map is clearly linear, and by the following it is also continuous

$$\begin{aligned}\|L^{-1}(f, \varphi)\|_{C^{2,\mu}(\bar{\Omega})} &= |u|_{2,\,\mu;\,\bar{\Omega}} \leq C(|f|_{0,\,\mu;\,\bar{\Omega}} + |\tilde{\varphi}|_{2,\,\mu;\,\bar{\Omega}}) \\ &\leq \tilde{C}(|f|_{0,\,\mu;\,\bar{\Omega}} + \|\varphi\|_{C^{2,\mu}(\partial\Omega)}) \leq \tilde{C}\|(f, \varphi)\|_{C^{0,\mu}(\bar{\Omega})\times C^{2,\mu}(\partial\Omega)} \,.\end{aligned}$$

□

Similarly, by differentiating the equation and induction one can also show existence and uniqueness in $C^{m,\mu}(\bar{\Omega})$.

Theorem 3.3.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ of class $C^{m,\mu}$ for some $m \geq 2$. Let Lu be as in (3.27) with $c \leq 0$, $a^{ij} \geq \lambda \cdot \delta^{ij}$ for some $\lambda > 0$ and $|a^{ij}, b^i, c|_{m-2,\,\mu;\,\bar{\Omega}} \leq \Lambda$. Then the operator*

$$L\colon C^{m,\mu}(\bar{\Omega}) \to C^{m-2,\mu}(\bar{\Omega}) \times C^{m,\mu}(\partial\Omega), u \mapsto (Lu, u|_{\partial\Omega})$$

is an isomorphism of Banach spaces.

Proof. To argue as in the previous proof, we only need higher order estimates. We assume that L and u are as stated above and $m = 3$, i.e. $u \in C^{3,\mu}(\bar{\Omega})$. Note that by differentiating equation (3.27) in direction e_l we obtian that $v = D_l u$ satisfies

$$Lv = D_l f - \sum_{|\alpha|,|\beta|\leq 1} (D_l a_{\alpha\beta}) D^{\alpha+\beta} u =: \tilde{f} \in C^{0,\mu}(\bar{\Omega})$$

with $|\tilde{f}|_{0,\,\mu;\,B_R(x)\cap\Omega} \leq |f|_{1,\,\mu;\,B_R(x)\cap\Omega} + C\,|u|_{2,\,\mu;\,B_R(x)\cap\Omega}$. So in the interiour, i.e. for $B_R(y) \subset \Omega$ we obtain by summation over l and the interior Schauder estimates, Theorem 3.2.2, and interpolation that

$$|u|_{3,\mu;\,B_{\theta R}(y)} \leq C(|u|_{2,\,\mu;\,B_R(y)} + |f|_{1,\,\mu;\,B_R(y)}) \,.$$

For estimates at the boundary, we can consider again $\tilde{u} := u - \varphi$ to restrict to the case that $u = 0$ on $\partial\Omega$. By flattening out the boundary locally we can assume that $B_R(x_0) \cap \Omega = B_R(0) \cap \mathbb{R}^n_+$ for $x_0 \in \partial\Omega$. For $l = 1, \dots, n-1$ we can argue as above, together with Theorem 3.3.1 to see that

$$|D^2 D_l u|_{0,\mu;\, B_{\theta R} \cap \mathbb{R}^n_+} \leq C(|u|_{2,\mu;\, B_R \cap \mathbb{R}^n_+} + |f|_{1,\mu;\, B_R \cap \mathbb{R}^n_+}) \,, \tag{3.28}$$

for $l = 1, \dots n-1$ and for $\theta \in (0, \theta_0)$ where $\theta_0 \in (0,1]$. Thus we have estimates for $|D_i D_j D_l u|_{0,\mu;\, B_{\theta R} \cap \mathbb{R}^n_+}$ for $i, j \in \{1, \dots, n\}$ and $l \in \{1, \dots, n-1\}$. Thus we only need to estimate $|D_n D_n D_n u|_{0,\mu;\, B_{\theta R} \cap \mathbb{R}^n_+}$. But note that $D_n u$ satisfies

$$a^{ij} D_{ij}(D_n u) + b^i D_i(D_n u) + c D_n u = \tilde{f}$$

as above, and thus

$$D_n D_n D_n u = \frac{1}{a_{nn}} \Big(- \sum_{\substack{i,j \\ (i,j) \neq (n,n)}} a^{ij} D_{ij}(D_n u) - b^i D_i(D_n u) - c D_n u + \tilde{f} \Big) \,.$$

This implies that also

$$|D_n D_n D_n u|_{0,\mu;\, B_{\theta R} \cap \mathbb{R}^n_+} \leq C(|u|_{2,\mu;\, B_R \cap \mathbb{R}^n_+} + |f|_{1,\mu;\, B_R \cap \mathbb{R}^n_+}) \,.$$

Arguing as in Theorem 3.3.3 we obtain

$$|u|_{3,\mu;\,\Omega} \leq C(|u|_{2,\mu;\,\Omega} + |\varphi|_{3,\mu;\,\Omega} + |f|_{1,\mu;\,\Omega}) \leq C(|u|_{0;\,\Omega} + |\varphi|_{3,\mu;\,\Omega} + |f|_{1,\mu;\,\Omega}) \,,$$

where we used the global Schauder estimates, Theorem 3.3.3, in the last step to estimate $|u|_{2,\mu;\,\Omega}$. Higher order estimates follow inductively. To get the existence of solutions in $C^{m,\mu}(\Omega)$ one argues as in the proof of Theorem 3.3.5. The rest follows as in the previous theorem. □

Lemma 3.3.8. *Let $B \subset \mathbb{R}^n$ be a ball and $\varphi \in C^0(\partial B)$. Assume L is uniformly elliptic with $|a_{\alpha\beta}|_{m,\mu;\,B} < \Lambda$, $|f|_{m,\mu;\,B} < \infty$, $c \leq 0$. Then the Dirichlet Problem*

$$\begin{aligned} Lu &= f \qquad \text{in } B \\ u &= \varphi \qquad \text{on } \partial B \end{aligned}$$

has a unique solution in $C^{m+2,\mu}(B) \cap C^0(\bar{B})$.

Proof. Approximate φ in $C^0(\partial B)$ by C^∞-functions φ_k and extend these φ_k to $\tilde{\varphi}_k$ on B. By Theorem 3.3.7 there exist unique solutions $u_k \in C^{m+2,\mu}(\bar{B})$ of $Lu_k = f$ on B, with $u_k = \tilde{\varphi}_k$ on ∂B. By the maximum principle the sequence (u_k) converges $C^0(\bar{B})$ to a function u. From the interior estimates in the proof of Theorem 3.3.7 the sequence (u_k) is uniformly bounded in $C^{m+2,\mu}(K)$ for any $K \Subset B$. Thus $u \in C^{m+2,\mu}(B)$ and $Lu = f$. By the maximum principle u is unique. □

Theorem 3.3.9. *Let $u \in C^2(\Omega)$ be a solution of $Lu = f$, where L is uniformly elliptic with $a_{\alpha\beta}, f \in C^{k,\mu}(\Omega)$. Then $u \in C^{k+2,\mu}(\Omega)$.*

Proof. It suffices to prove this for arbitrary balls $B \Subset \Omega$. We first start with $k = 0$. Let B be such a ball and consider the Dirichlet problem for v:

$$L_0 v := a^{ij} D_{ij} v + b^i D_i v = f - cu =: \tilde{f}\,, \qquad v = u \text{ on } \partial B\,.$$

Since $u \in C^2(\bar{B})$ it follows that $\tilde{f} \in C^{0,\mu}(\bar{B})$. Thus by Lemma 3.3.8 a solution v exists in $C^{2,\mu}(B) \cap C^0(\bar{B})$. By the maximum principle we have $u \equiv v$ and thus $u \in C^{2,\mu}(B)$. Since this holds for any $B \Subset \Omega$ we have $u \in C^{2,\mu}(\Omega)$. Then proceed by induction. □

Remark 3.3.10: By Corollary 3.8 in [3], which is a corollary to Theorem 3.1.4 in these notes, solutions to $Lu = f$ are unique in sufficiently small domains, even if we don't assume $c \leq 0$. This makes it possible to prove the theorem above including the boundary, provided $\partial\Omega$ is of class $C^{k,\mu}$. See exercises.

We will now collect the corresponding estimates and theorems in the case that the operator $\tilde{L}$ is in divergence form, i.e. $u \in C^1(\Omega)$ solves weakly

$$\tilde{L}u = D_i(a^{ij} D_j u + b^i u) + c^i D_i u + du = D_i f^i + g\,, \tag{3.29}$$

which we also write in multi-index notation as

$$\sum_{|\alpha|,|\beta|\leq 1} D^\beta(a_{\alpha\beta} D^\alpha u) = \sum_{|\beta|\leq 1} D^\beta f_\beta\,.$$

Recall that in Theorem 3.2.1 we have shown interior $C^{1,\mu}$-estimates for solutions of (3.29). Now it is straightforward to obtain local boundary estimates in this case as in Theorem 3.3.1. Here the boundary is only needed to be of the class $C^{1,\mu}$. Together with the interior estimate we obtain the following global estimates:

Theorem 3.3.11. *Let $u \in C^{1,\mu}(\bar{\Omega})$ be a weak solution of (3.29) on a domain Ω with $\partial\Omega$ of class $C^{1,\mu}$, where $\tilde{L}$ is uniformly elliptic and $|a_{\alpha\beta}|_{0,\,\mu;\,\Omega} \leq \Lambda$. Furthermore let $u = \varphi$ on $\partial\Omega$, where $\varphi \in C^{1,\mu}(\bar{\Omega})$ and $g, f^i \in C^\mu(\bar{\Omega})$. Then*

$$|u|_{1,\,\mu;\,\bar{\Omega}} \leq C(|u|_{0;\,\bar{\Omega}} + |\varphi|_{1,\,\mu;\,\bar{\Omega}} + |g|_{0,\,\mu;\,\bar{\Omega}} + |f|_{0,\,\mu;\,\bar{\Omega}})\,,$$

where $C = C(n, \lambda, \Lambda, \Omega)$.

We also obtain the following existence result:

Theorem 3.3.12. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ of class $C^{1,\mu}$, and let the operator $\tilde{L}$ as in (3.29) be uniformly elliptic and $|a_{\alpha\beta}|_{0,\,\mu;\,\Omega} \leq \Lambda$. Let $g, f^i \in C^\mu(\bar{\Omega})$ and $\varphi \in C^{1,\mu}(\bar{\Omega})$. Then the Dirichlet problem*

$$\tilde{L}u = g + D_i f^i \quad \text{in } \Omega\,, \;\; u = \varphi \quad \text{on } \partial\Omega$$

is uniquely solvable in $C^{1,\mu}(\bar{\Omega})$, *provided*

$$\int_\Omega (d\phi - b^i D_i\phi)\, dx \le 0 \tag{3.30}$$

for all $\phi \in C^1_c(\Omega)$, $\phi \ge 0$.

Remark 3.3.13: The condition (3.30) correponds to the case that for the equation in non-divergence form we have $c \le 0$. More precisely for any function $u \in W^{1,2}(\Omega)$, which satisfies $\tilde{L}u \ge 0$ in the weak sense, where $\tilde{L}$ is uniformly elliptic with bounded, measurable coefficients satisfying (3.30), it holds that $\sup_\Omega u \le \sup_{\partial\Omega} u^+$. See Theorem 8.1 in [3] and exercises.

Proof. W.l.o.g. we can assume $\varphi \equiv 0$. We approximate Ω from the inside by a sequence of $C^{2,\mu}$-domains Ω_k analogously as in Remark 3.3.4. As in the proof of Theorem 3.3.5 we can approximate by mollification in $C^0_{\text{loc}}(\Omega)$ the coefficients such that $a^k_{\alpha\beta}, f^k_\beta \in C^\infty(\bar{\Omega}_k)$ and

$$a^k_{\alpha\beta} \to a_{\alpha\beta}\,, \quad f^k_\beta \to f_\beta$$

in $C^0_{\text{loc}}(\Omega)$ with $|a^k_{\alpha\beta}|_{0,\mu;\Omega_k} \le \Lambda$ and $|f^k_\beta|_{0,\mu;\Omega_k} \le |f_\beta|_{0,\mu;\Omega}$. We can furthermore assume that $a^{ij}_k \ge \lambda \cdot \delta^{ij}$ and that d^k, b^i_k satisfy (3.30). By Theorem 3.3.5 there exist $C^{2,\mu}$-solutions u_k to

$$\tilde{L}_k u_k = g_k + D_i f^i_k \quad \text{in } \Omega\,,\ u_k = 0 \quad \text{on } \partial\Omega_k\ ,$$

which satisfy the global estimates of Theorem 3.3.11, where the constant C can be chosen to be indendent of k. By an a-priori sup-bound for weak solutions of (3.29), see [3], Theorem 8.16, we have

$$\sup_{\Omega_k} |u_k| \le C \sup_{\Omega_k} |f^k_\beta|\,.$$

This implies the uniform estimate

$$|u_k|_{1,\mu;\Omega_k} \le C|f^k_\beta|_{0,\mu;\Omega_k} \le C|f^k_\beta|_{0,\mu;\Omega}\ ,$$

with C independent of k. Using Arzela-Ascoli and the weak formulation of (3.29) we obtain a solution $u \in C^{1,\mu}(\bar{\Omega})$ of our inital problem. By the maximum principle for weak solutions, see Remark 3.3.13, this solution is unique in the class of $W^{1,2}$-solutions. □

4 Quasi-linear Equations

4.1 Some Applications to quasi-linear Equations

Let M^n be a hypersurface, which is given locally as the graph of a funtion u, that is for all $x_0 \in M^n$ there is an $R > 0$, such that after possibly rotating the coordinate system and translating x_0 to 0:

$$M^n \cap B_R(0) = \big\{x \in \mathbb{R}^{n+1} \,\big|\, x_{n+1} = u(x_1, \ldots, x_n)\big\} \cap B_R(0),$$

where $u : B_R^n(0) = \big\{(x_1, \ldots, x_n) \,\big|\, \exists\, x_{n+1} : (x_1, \ldots, x_n, x_{n+1}) \in B_R^{n+1}(0)\big\} \to \mathbb{R}$ and $u \in C^2\big(B_R^n(0)\big)$. The mean curvature H of M^n is then given by

$$H = \mathcal{A}u = D_i\left(\frac{D_i u}{\sqrt{1 + |Du|^2}}\right) = a^{ij}(Du) D_i D_j u$$

with

$$a^{ij}(Du) = \frac{1}{\sqrt{1 + |Du|^2}}\left(\delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}\right).$$

Remark 4.1.1: i) If $|Du| < C$ in $B_R(0)$, then (a^{ij}) is uniformly elliptic:

$$\begin{aligned}
a^{ij}(Du)\xi_i\xi_j &= \frac{1}{\sqrt{1 + |Du|^2}}\left(|\xi|^2 - \frac{\langle Du\,,\,\xi\rangle^2}{1 + |Du|^2}\right) \\
&\geq \frac{1}{\sqrt{1 + |Du|^2}}\left(|\xi|^2 - \frac{|Du|^2|\xi|^2}{1 + |Du|^2}\right) \\
&= \frac{1}{\sqrt{1 + |Du|^2}}\left(\frac{1}{1 + |Du|^2}\right)|\xi|^2 \\
&\geq \frac{1}{(1 + C^2)^{3/2}}|\xi|^2
\end{aligned}$$

ii) If $M^n \cap B_R(0)$ is only representable as a C^1-graph (resp. a $W^{1,2}$-graph) then we understand

$$\mathcal{A}u = \mathcal{H}(x, u)$$

as

$$-\int\limits_{B_R(0)} \frac{1}{\sqrt{1+|Du|^2}} D_i u D_i \varphi \, dx^n = \int \mathcal{H}(x,u)\varphi \, dx^n \qquad \forall \varphi \in C_0^1\big(B_R^n(0)\big)$$

for a given function $\mathcal{H}(x,u)$ on $B_R^n(0) \times \mathbb{R}$.
In this weak setting the coefficients in front of the highest derivatives have the form

$$\tilde{a}^{ij}(x,u,Du) = \frac{1}{\sqrt{1+|Du|^2}} \delta^{ij}$$

and again uniform ellipticity follows from $|Du| \le C$, that is there exist $\lambda, \Lambda > 0$, s.t.

$$\lambda |\xi|^2 \le \tilde{a}^{ij}(x,u,Du)\xi_i \xi_j \le \Lambda |\xi|^2.$$

iii) Let M^n be a C^1-hypersuface. Assume that it fulfils the equation

$$\int\limits_{M^n} \operatorname{div}_{M^n}(X) d\mu = \int\limits_{M^n} \Big\langle X \,,\, \tilde{H} \Big\rangle \, d\mu \tag{4.1}$$

for all $X \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, and $\tilde{H} = \bar{H} \cdot \nu$, where $\bar{H}$ is a function on M^n and ν a choice of a unit normal vector field on M^n. Here the divergence on M^n is defined via

$$\big(\operatorname{div}_{M^n}(X)\big)(x_0) = \sum_{i=1}^{n} \langle D_{e_i} X \,,\, e_i \rangle \, (x_0) \,,$$

where e_i is an orthonormal basis of the tangent space $T_{x_0} M^n$.

If M^n is locally representable as the C^2-graph of a function u, then we can test (4.1) with vectorfields $X = \varphi \cdot \nu$, with $\varphi : B_R^n(0) \to \mathbb{R}$ and $\varphi \in C_c^1\big(B_R^n(0)\big)$. Since $u \in C^2$ we have $\nu \in C^1$ and $\varphi \cdot \nu \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Then we have

$$\operatorname{div}_{M^n}(\varphi \cdot \nu) = \underbrace{\langle \nabla_{M^n} \varphi \,,\, \nu \rangle}_{=0} + \varphi \cdot \operatorname{div}_{M^n}(\nu),$$

where we have extended ν to a vectorfield on $\mathbb{R}^{n+1} \cap B_R^n$ by

$$\nu(x_1, \dots, x_n, x_{n+1}) = \frac{1}{\sqrt{1+|Du|^2(x_1,\dots,x_n)}} (Du(x_1,\dots,x_n), -1).$$

As well it holds

$$\begin{aligned}
\operatorname{div}_{M^n}(X) &= \operatorname{div}_{\mathbb{R}^{n+1}}(X) - \langle D_\nu X\,,\,\nu\rangle \\
\operatorname{div}_{M^n}(\nu) &= \operatorname{div}_{\mathbb{R}^{n+1}}(\nu) - \underbrace{\langle D_\nu \nu\,,\,\nu\rangle}_{=0,\text{ since }\langle \nu\,,\,\nu\rangle = 1} \\
\Longrightarrow \operatorname{div}_{M^n}(\nu) &= \operatorname{div}_{\mathbb{R}^{n+1}}(\nu) \\
&= \operatorname{div}_{\mathbb{R}^{n+1}}\left(\frac{1}{\sqrt{1+|Du|^2}}(Du,-1)\right) \\
&= \operatorname{div}_{\mathbb{R}^{n}}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \qquad \text{since independent of } x_{n+1} \\
&= \mathcal{A}u.
\end{aligned}$$

Putting this in (4.1) we see that

$$\int\limits_{B_R^n} \varphi \mathcal{A}u \sqrt{1+|Du|^2}\,dx^n = \int\limits_{B_R^n} \varphi \bar{H} \sqrt{1+|Du|^2}\,dx^n \qquad \forall\,\varphi \in C_c^1(B_R^n(0)).$$

Replace $\varphi \mapsto \frac{\varphi}{\sqrt{1+|Du|^2}}$ and obtain

$$\int\limits_{B_R^n} \varphi \mathcal{A}u\,dx^n = \int\limits_{B_R^n} \varphi \bar{H}\,dx^n$$

$$\Longrightarrow \mathcal{A}u = \bar{H}(x,u).$$

Thus, if (4.1) holds for all vectorfields of the form $X = \varphi\nu$ and $\tilde{H} = \bar{H}\nu$, then the mean curvature is given by $\mathcal{A}u = \bar{H}$. Note that (4.1) is well-defined if M^n is only in C^1. Approximating M^n locally by C^2 graphs we see that it still holds that in this case

$$\int\limits_{B_R(0)} \frac{1}{\sqrt{1+|Du|^2}} D_i u D_i \varphi\,dx^n = -\int \varphi \bar{H}(x,u(x))\,dx^n \qquad \forall \varphi \in C_0^1\big(B_R^n(0)\big)$$

Theorem 4.1.2. *Let $M^n \subset \mathbb{R}^{n+1}$ be a $C^{1,\alpha}$-hypersurface, that is, it is locally representable as a $C^{1,\alpha}$-graph. Assume that M^n has constant mean curvature H_0 as defined in* (4.1)*. Then M^n is smooth.*

Remark 4.1.3: We will see later that M^n in C^1 and H_0 smooth suffices. By Remark 4.1.1 the graph function u fulfills the PDE

$$D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = H_0, \qquad u \in C^{1,\alpha}\big(B_R(0)\big) \tag{4.2}$$

in the weak sense.

We first need the following lemma:

Lemma 4.1.4. *Let $u \in C^1\big(B_R(0)\big)$ be a weak solution of (4.2). Then $u \in C^1 \cap W^{2,2}_{loc}(B_R)$ and $w = D_l u$ is a weak solution of*

$$D_i(a^{ij} D_j w) = 0 \qquad om\ B_R(0) =: \Omega, \tag{4.3}$$

with

$$a^{ij}(x) = \frac{\delta^{ij}}{\sqrt{1+|Du|^2}} - \frac{D^i u(x) \cdot D^j u(x)}{(1+|Du|^2(x))^{3/2}}.$$

for $l = 1, \dots, n$.

Proof. We use difference quotients: Let

$$\tau_{h,s} u(x) := \frac{1}{h}\big(u(x+he_s) - u(x)\big)$$

and

$$A^i(p) := \frac{p_i}{\sqrt{1+|p|^2}}, \qquad p \in \mathbb{R}^n.$$

With this (4.2)can be written as

$$D_i\big(A^i(Du)\big) = H_0. \tag{4.4}$$

For difference quotients we have the following 'partial integration' rule (see [2], 5.8.2 proof of Theorem 3),

$$\int f \tau_{-h} g \, dx = -\int g \tau_h f \, dx, \qquad f \text{ or } g \in C^0_c(\Omega),\ h \text{ small enough.} \tag{4.5}$$

Choose $\eta \in C^\infty_c(\Omega)$ and let $|h| < \frac{1}{2} \operatorname{dist}\big(\partial\Omega, \operatorname{supp}(\eta)\big)$. We test (4.4) mit $\varphi := \tau_{-h,l}(\eta^2 \tau_{h,l} u)$. We will in the following supress the subscript l.

$$-\int_\Omega \big(A^i(Du)\big) D_i \varphi = \int_\Omega H_0 \varphi.$$

The right hand side vanishes since

$$\int_\Omega H_0 \varphi \, dx = \int_\Omega H_0 \tau_{-h}(\eta^2 \tau_h u) \, dx \overset{(4.5)}{=} -\int_\Omega \underbrace{\tau_h H_0}_{=0} (\eta^2 \tau_h u) \, dx = 0.$$

Using that D_j and τ_h commute, we obtain

$$\int \tau_h A^i(Du) D_i(\eta^2 \tau_h u) \, dx = 0. \tag{4.6}$$

Furthermore

$$\tau_h A^i(Du) = \frac{1}{h}\int_0^1 \frac{d}{dt}\big(A^i(Du + th\tau_h Du)\big)\,dt = \Theta^{ij} D_j \tau_h u$$

$$\begin{aligned} \text{with } \Theta^{ij} &= \int_0^1 \frac{\partial A^i}{\partial p^j}(Du + th\tau_h Du)\,dt \\ &= \int_0^1 a^{ij}(Du + th\tau_h Du)\,dt \\ &= \int_0^1 a^{ij}\Big(Du + t\big(Du(x+he_i) - Du(x)\big)\Big)\,dx\,. \end{aligned}$$

Since

$$\begin{aligned} \eta D_j(\tau_h u) &= D_j(\eta\tau_h u) - (\tau_h u) D_j(\eta) \qquad \text{and} \\ D_i(\eta^2 \tau_h u) &= \eta(\tau_h u) D_i \eta + \eta D_i(\eta \tau_h u) \end{aligned}$$

we see from (4.6) that

$$\int \Theta^{ij} D_i(\eta\tau_h u) D_j(\eta\tau_h u)\,dx = \int \Theta^{ij} D_i\eta D_j \eta (\tau_h u)^2\,dx\,.$$

We can compute

$$a^{ij}(p) = \frac{\partial A^i}{\partial p_j} = \frac{1}{\sqrt{1+|p|^2}}\left(\delta^{ij} - \frac{p^i p^j}{1+|p|^2}\right).$$

Since $u \in C^1(\Omega)$ we have $|Du| \le C$. Thus

$$\Theta^{ij} \ge \lambda\delta^{ij}, \quad \lambda > 0 \qquad \text{und} \qquad \Theta^{ij} \le \Lambda\delta^{ij}, \quad \Lambda > 0 \qquad .$$

$$\Longrightarrow \int_\Omega |D(\eta v)|^2\,dx \le C\int_\Omega |D\eta|^2 v^2\,dx$$

mit $v := \tau_h u$. Choose $\Omega' \Subset \Omega$ and let $\eta \in C_c^\infty(\Omega)$ with $\eta \equiv 1$ on Ω' and $0 \le \eta \le 1$. This implies

$$\int_{\Omega'} |Dv|^2 \le C,$$

independent of h, since $|v| = |\tau_h u| \le C'$ uniformly in h. As in PDE I (see [2], 5.8.2, Theorem 3) this implies for $h \to 0$ that

$$u \in W^{2,2}_{loc}(\Omega).$$

That $D_l u$ solves (4.3) follows by choosing the testfunction $\eta = D_l\psi$, $\psi \in C_c^\infty(\Omega)$ in (4.4) and then integrating by parts. □

Proof of Theorem 4.1.2. Note that the coefficients a^{ij} are lipschitz-continous in Du. Thus, since $u \in C^{1,\alpha}(\bar{B}_R)$ we have $a^{ij} \in C^{0,\alpha}(\bar{B}_R)$. For $w := D_\ell u$ let w_k be a sequence of $C^\infty(\bar{B}_R)$-functions with $w_k \to w$ in $C^{0,\alpha}(\bar{B}_R)$. By Theorem 3.3.12 there is a unique solution $v_k \in C^{1,\alpha}(\bar{B}_R)$ of the Dirichlet-problem

$$\begin{cases} D_i(a^{ij} D_j v_k) = 0 & \text{on } B_R \\ v_k = w_k & \text{on } \partial B_R \end{cases}$$

and the solutions satisfy

$$\|v_k\|_{C^{1,\alpha}(B_{\theta R})} \leq c \cdot \sup_{B_R} |v_k| \leq c \sup_{\partial B_R} |w_k| \leq c \sup_{\partial B_R} |w|$$

for any $\theta \in (0,1)$. By the weak maximum principle for weak solutions, see Remark 3.3.13, the sequence converges in $C^0(\bar{B}_R)$ to a function v. Since the sequence is uniformly bounded in $C^{1,\alpha}_{\text{loc}}(B_R)$ we have that $v \in C^{1,\alpha}_{\text{loc}}(B_R) \cap C^0(\bar{B}_R)$ and v solves

$$\begin{cases} D_i(a^{ij} D_j v) = 0 & \text{on } B_R \\ v = w & \text{on } \partial B_R \end{cases}$$

Again by Remark 3.3.13 we have $v \equiv w$ on B_R. Thus

$$w \in C^{1,\alpha}_{\text{loc}}(B_R) \Rightarrow u \in C^{2,\alpha}_{\text{loc}}(B_R) \Rightarrow a_{ij} \in C^{1,\alpha}_{\text{loc}}(B_R) \Rightarrow u \in C^{3,\alpha}_{\text{loc}}(B_R) \Rightarrow \cdots \Rightarrow u \in C^\infty$$

□

Remark 4.1.5: Theorem 4.1.2 is still true if one replaces $H_0 = \text{const.}$ by $H = \mathcal{H}(x, u, Du)$ where $\mathcal{H} \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $\mathcal{H} = \mathcal{H}(x, z, p)$ on $\{(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \mid (x, z, p) = (x, u(x), Du(x))\}$

Note that the assumption implies that

$$\left|\frac{\partial \mathcal{H}}{\partial x}\right|, \left|\frac{\partial \mathcal{H}}{\partial p}\right|, \left|\frac{\partial \mathcal{H}}{\partial z}\right| \leq C$$

on this set, and the same holds for all higher partial derivatives. See exercises.

4.2 Existence of solutions to quasi-linear equations

As an example, but as well as as guiding principle in the general case we will study the following problem. Let $\Omega \subset \mathbb{R}^n$ be a domain and study the problem

$$\begin{cases} \mathcal{A}u = \mathcal{H}(x, u) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

This is known as the prescribed mean curvature equation for graphical solutions.

Theorem 4.2.1 (Existence of solutions to (4.7), given $C^{1,\alpha'}$ a-priori bounds)**.**
Let $\Omega \subset \mathbb{R}^n$ be a domain, $\partial\Omega$ of the class $C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. If $\mathcal{H} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is of the class C^1 with $\frac{\partial}{\partial t}\mathcal{H}(x,t) \geq 0 \ \forall (x,t) \in \Omega \times \mathbb{R}$, $\frac{\partial}{\partial t}\mathcal{H}(x,t)$ of class $C^{0,\alpha}$, and there are positive constants $C' = C'(\Omega, \mathcal{H}, \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})})$, $\alpha' = \alpha'(\Omega, \mathcal{H}, \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})})$, such that for every solution $v \in C^{2,\alpha}(\bar{\Omega})$ of (4.7) it holds:

$$\|v\|_{C^{1,\alpha'}(\overline{\Omega})} \leq C' \,, \tag{4.8}$$

for some $0 < \alpha' < 1$, then (4.7) has a solution for all $\varphi \in C^{2,\alpha}(\overline{\Omega})$.

We want to apply the continuity method. For that we study the following family of Dirichlet problems. For $t \in [0,1]$ consider

$$\begin{cases} \mathcal{A}u_t = t\mathcal{H}(x,u_t) & \text{in } \Omega \\ u_t = t\varphi & \text{on } \partial\Omega \end{cases} \tag{4.9}$$

For $t = 0$ we have the trivial solution $u_0 \equiv 0$. We first need the following lemma.
Lemma 4.2.2 (Uniqueness of solutions to (4.9))**.** *Assume $v_1, v_2 \in C^2(\Omega)$ solve*

$$\mathcal{A}v_1 = t\mathcal{H}(x,v_1) \ \text{ in } \Omega \qquad\qquad v_1 = t\varphi \ \text{ on } \partial\Omega$$
$$\mathcal{A}v_2 = t\mathcal{H}(x,v_2) \ \text{ in } \Omega \qquad\qquad v_2 = t\varphi \ \text{ on } \partial\Omega$$

then $v_1 = v_2 \quad$ on Ω

Proof. We have

$$(\mathcal{A}v_1 - \mathcal{A}v_2)(v_1 - v_2) = t(\mathcal{H}(x,v_1) - \mathcal{H}(x,v_2))(v_1 - v_2) \,.$$

Integrating this equation gives

$$-\int\limits_{\Omega} (A^i(Dv_1) - A^i(Dv_2))D_i(v_1 - v_2)\,dx = t\int\limits_{\Omega} \big(\mathcal{H}(x,v_1) - \mathcal{H}(x,v_2)\big)(v_1 - v_2)\,dx \,.$$

We can also compute

$$A^i(Dv_1) - A^i(Dv_2) = \int\limits_0^1 \frac{d}{d\tau}(A^i(\tau Dv_1 + (1-\tau)Dv_2))\,d\tau$$
$$= \int\limits_0^1 \frac{\partial A^i}{\partial p_j}(\tau Dv_1 + (1-\tau)Dv_2)D_j(v_1 - v_2)\,d\tau$$
$$= \int\limits_0^1 a^{ij}(\tau Dv_1 + (1-\tau)Dv_2)\,d\tau \cdot D_j(v_1 - v_2) =: \Theta^{ij}D_j(v_1 - v_2).$$

Similarly one obtains

$$\mathcal{H}(x,v_1)-\mathcal{H}(x,v_2)=\int\limits_0^1 \frac{d}{d\tau}\mathcal{H}(x,\tau v_1+(1-\tau)v_2)\,d\tau$$
$$=\int\limits_0^1 \left(\frac{\partial}{\partial z}\mathcal{H}\right)(x,\tau v_1+(1-\tau)v_2)\,d\tau\cdot(v_1-v_2)\,.$$

Furthermore it follows that

$$a^{ij}(p)\geq \frac{1}{(1+|p|^2)^{3/2}}\delta_{ij} \quad\Rightarrow\quad \Theta^{ij}\geq \frac{1}{(1+\sup_\Omega(|Dv_1|+|Dv_2|))^{3/2}}\delta^{ij}=:\lambda\cdot\delta^{ij}$$

and $\frac{\partial}{\partial z}\mathcal{H}\geq 0$ implies

$$\int\limits_\Omega \Theta^{ij}D_j(v_1-v_2)D_i(v_1-v_2)\,dx=$$
$$-t\int\limits_\Omega\int\limits_0^1\left(\frac{\partial}{\partial z}\mathcal{H}\right)(x,\tau v_1+(1-\tau)v_2)\,d\tau\cdot(v_1-v_2)^2dx\leq 0\,.$$

But this implies

$$\int\limits_\Omega \Theta_{ij}D_j(v_1-v_2)D_i(v_1-v_2)dx\geq\lambda\int\limits_\Omega |D(v_1-v_2)|^2dx$$
$$\Rightarrow\int\limits_\Omega |D(v_1-v_2)|^2dx=0\Rightarrow v_1-v_2=const.\Rightarrow v_1\equiv v_2$$

since $v_1=v_2$ on $\partial\Omega$. □

Remark 4.2.3: If we have a general quasi-linear equation of the form

$$Qu=a^{ij}(x,u,Du)D_{ij}u+b(x,u,Du)$$

with $\lambda\delta^{ij}\leq a^{ij}\leq\Lambda\delta^{ij}$, one can show that the difference of two solutions fulfills a linear equation of second order. One obtains:

Theorem 4.2.4. *Let $u,v\in(C^0(\overline{\Omega})\cap C^2(\Omega))$ with $Qu\geq Qv$ in Ω, $u\leq v$ on $\partial\Omega$ and*

1. *Q is uniformly elliptic with respect to either u or v.*
2. *The coefficients a^{ij} are independent of z.*
3. *The coefficient b is non-increasing in z $\forall(x,p)\in\Omega\times R^n$.*
4. *The coefficients a^{ij}, b are continuously differentiable with respect to the p-variable in $\Omega\times R\times R^n$.*

Then $u \le v$ *in* Ω.

For a proof see Theorem 10.1 in [3], or the exercises.

Proof of Theorem 4.2.1. Let

$$S := \{t \in [0,1] : (4.9) \text{ has a unique solution } u_t \in C^{2,\alpha}(\overline{\Omega})\} .$$

Since $0 \in S$ (see the remark after (4.9)) it suffices to show that, that S is open and closed.

(i) S is closed: Let $t_n \to t_0$, $t_n \in S$. Since $t_n \in S$, there exits $u_{t_n} \in C^{2,\alpha}(\overline{\Omega})$, such that

$$\mathcal{A}u_{t_n} = t_n \mathcal{H}(x, u_{t_n}) \text{ in } \Omega$$
$$u_{t_n} = t_n \varphi \text{ on } \partial\Omega$$

and $\|u_{t_n}\|_{C^{1,\alpha}(\overline{\Omega})} \le C(\|t_n\varphi\|_{C^{2,\alpha}(\overline{\Omega})} + \|t_n\mathcal{H}\|_{C^1(\Omega\times\mathbb{R})}) \le C_0$, independent of u. We have

$$\mathcal{A}u_{t_n} = a^{ij}(Du_{t_n})D_{ij}u_{t_n} = t_n\mathcal{H}(x, u_{t_n}).$$

Let $a^{ij}(Du_{t_n}(x)) =: \tilde{a}_n^{ij}(x), f_n(x) := t_n\mathcal{H}(x, u_{t_n})$. Then $\|\tilde{a}_n^{ij}\|_{C^{0,\alpha'}(\overline{\Omega})}, \|f_n\|_{C^{0,\alpha'}(\overline{\Omega})} \le C_0$ independent of u_{t_n}. Furthermore, u_{t_n} solves the linear equation

$$\begin{cases} \tilde{a}_n^{ij} D_{ij} u_{tn} = f_n(x) & \text{in } \Omega \\ u_{t_n} = t_n\varphi & \text{on } \partial\Omega \end{cases}$$

Take $\beta = \min\{\alpha, \alpha'\}$. Using the global Schauder estimates we see

$$\|u_{t_n}\|_{C^{2,\beta}(\overline{\Omega})} \le C(\|u_{t_n}\|_{C^0(\overline{\Omega})} + \|f_n\|_{C^{0,\beta}(\overline{\Omega})} + \|\varphi\|_{C^{2,\beta}(\overline{\Omega})}) \le \tilde{C} .$$

If $\alpha' < \alpha$, we see that this implies that now $\|\tilde{a}_n^{ij}\|_{C^{0,\alpha}(\overline{\Omega})}, \|f_n\|_{C^{0,\alpha}(\overline{\Omega})} \le C_0$ independent of u_{t_n} and thus again

$$\|u_{t_n}\|_{C^{2,\alpha}(\overline{\Omega})} \le C(\|u_{t_n}\|_{C^0(\overline{\Omega})} + \|f_n\|_{C^{0,\alpha}(\overline{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\overline{\Omega})}) \le \tilde{C} .$$

Now use Arzela-Ascoli to extract a convergent subsequence in $C^2(\overline{\Omega})$, which converges to a solution u of

$$\begin{cases} \mathcal{A}u = t_0\mathcal{H}(x, u) & \text{in } \Omega \\ u = t_0\varphi & \text{on } \partial\Omega \end{cases}$$

Furthermore, $u \in C^{2,\alpha}(\bar{\Omega})$ and thus $t_0 \in S$.

(ii) S is open: Let $t_0 \in S$ and $u_{t0} \in C^{2,\alpha}(\overline{\Omega})$ be a solution of

$$\begin{cases} \mathcal{A}u_{t0} = \mathcal{H}(x, u_{t0}) & \text{in } \Omega \\ u_{t0} = t_0\varphi & \text{on } \partial\Omega \end{cases}$$

We have to show: There exists $\delta > 0$ such that $\forall t \in (t_0 - \delta, t_0 + \delta)$ there exists a solution u_t of (4.9). We aim to apply the implicit function theorem on Banach spaces to the following function:

$$T : [0,1] \times C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega),\ (t,u) \mapsto (\mathcal{A}u - t\mathcal{H}(\cdot,u), u|_{\partial\Omega} - t\varphi)\,.$$

Note that we have $T(t_0, u_{t_0}) = (0,0)$. We aim to find a function $h : (t_0 - \delta, t_0 + \delta) \to C^{2,\alpha}(\overline{\Omega})$, such that $h(t_0) = u_0$, $T(t, h(t)) = 0\ \forall t \in (t_0 - \delta, t_0 + \delta)$. This implies the existence of a solution u_t for all $t \in (t_0 - \delta, t_0 + \delta)$. The existence of such a funtion h is guaranteed by the implicit function theorem ([1, chapter 10]), provided we can show that the Gateaux derivative with respect to the second variable of T is an isomorphism between Banach spaces. To compute the Gateaux derivative, we can compute the Frechet derivative, which is a directional derivative. If all the directional derivatives exist, as in the finite dimensional case, and are bounded operators, which depend continously on u_{t_0} in a neighbourhood of u_{t_0}, then the full derivative exists and it coincides with the directional derivatives. To compute the Frechet derivative, let $h \in C^{2,\alpha}(\overline{\Omega})$. We have

$$\begin{aligned}
&D_2T(t_0,u_0)(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} T(t_0, u_{t_0} + \epsilon h)\\
&= \frac{d}{d\epsilon}\Big|_{\varepsilon=0} (a^{ij}(D(u_{t_0} + \epsilon h))D_{ij}(u_{t_0} + \epsilon h) - t_0\mathcal{H}(\cdot, u_{t_0} + \epsilon h), u_{t_0} - \epsilon h|_{\partial\Omega} - t_0\varphi)\\
&= (a^{ij}(D(u_{t_0}))D_{ij}h + \frac{\partial a^{ij}}{\partial p^k}(Du_{t_0})D_k h D_{ij}u_{t_0} - t_0\frac{\partial\mathcal{H}}{\partial t}(\cdot, u_{t_0})h, h|_{\partial\Omega})
\end{aligned}$$

We define

$$\begin{aligned}
\tilde{a}^{ij}(x) &:= a^{ij}(Du_{t_0}(x))\\
\tilde{b}^k(x) &:= \frac{\partial a_{ij}}{\partial p_k}(Du_{t_0}(x))D_{ij}u_{t_0}(x)\\
\tilde{c}(x) &:= -t_0\frac{\partial\mathcal{H}}{\partial t}(x, u_{t_0}(x))\,.
\end{aligned}$$

Note $\tilde{c} \le 0$ and $\tilde{a}^{ij} \in C^{0,\alpha}(\overline{\Omega})$, $\tilde{b}^k \in C^{0,\alpha}(\overline{\Omega})$, $\tilde{c} \in C^{0,\alpha}(\overline{\Omega})$. Define

$$Lh := \tilde{a}^{ij}D_{ij}h + \tilde{b}^i D_i h + \tilde{c}h$$

which implies that

$$D_2T(t_0,u_0)(h) = (Lh, h_{|\partial\Omega})\,.$$

We see that L depends continously on u_{t_0} in $C^{2,\alpha}(\bar{\Omega})$. By Theorem 3.3.6 this is an isomorphism between Banach spaces, thus we can apply the implicit function theorem. This yields that S is open. □

To apply the continuity method it is essential that we have $c \le 0$ in the linearisation to be able to apply the implicit function theorem. There are further methods to prove the existence of solutions with the help of fixed point theorems, which we will present in the following.

Theorem 4.2.5 (Extension of Brouwer's fixed point theorem)**.** *Let C be a compact, convex set in a Banach space B and T a continuous map $T : C \to C$. Then T has a fixed point, that is $Tx = x$ for a $x \in C$.*

Proof. Let $k \in \mathbb{N}$. Since C is compact, there exist $x_1, \dots, x_N \in C$, such that $C \subset \bigcup_{i=1}^N B_{1/k}(x_i) =: \bigcup_{i=1}^N B_i$. Let $C_k \subset C$ be the konvex hull of $\{x_1, \dots, x_N\}$, i.e.

$$C_k = \left\{ \sum_{i=1}^N \lambda_i x_i, 0 \le \lambda_i \le 1, \sum_i \lambda_i = 1 \right\}$$

and define $J_k : C \to C_k$ by:

$$J_k(x) = \frac{\sum_i \operatorname{dist}(x, C \setminus B_i) x_i}{\sum_j \operatorname{dist}(x, C \setminus B_j)\,.}$$

J_k is continous and for all $x \in C$ we have:

$$\|J_k(x) - x\| \le \frac{\sum_i \big(\operatorname{dist}(x, C \setminus B_i) \|x_i - x\| \big)}{\sum_j \operatorname{dist}(x, C \setminus B_j)} \le \frac{1}{k}$$

Thus $J_k \circ T|_{C_k}$ maps C_k to C_k. Using Brouwer's fixed point theorem (C_k is homeomorphic to some B^n for some $n \in \mathbb{N}$) the map J_k has a fixed point x_k. Since C is compact, there is a subsequence $x_{k'} \to x \in C$. We have

$$\|x_{k'} - Tx_{k'}\| = \|J_k \circ Tx_{k'} - Tx_{k'}\| \le \frac{1}{k'}$$

and thus $x = Tx$. □

Corollary 4.2.6. *Let C be a closed, convex set in a Banach space B and T a continuous map from C to C, such that $T(C)$ is precompact, that is the closure of $T(C)$ is compact. Then T has a fixed point.*

Proof. See exercises. □

Theorem 4.2.7. *Let T be a compact map from a Banach space B onto itself. Assume further that there is a constant $M > 0$ such that*

$$\|x\|_B < M \quad \forall\, x \in B, s.t.\ x = \sigma T x \text{ for a } \sigma \in [0,1]. \tag{4.10}$$

Then T has a fixed point.

Proof. Define

$$T^*x := \begin{cases} Tx & \text{, if } \|Tx\| \le M \\ \frac{Tx}{\|Tx\|}M & \text{, if } \|Tx\| > M\,. \end{cases}$$

Thus $T^* : \overline{B}_M \to \overline{B}_M$ is continuous, and $T^*\overline{B}_M$ is precompact. Corollary 4.2.6 implies that T^* has a fixed point x. Assuming that $\|Tx\| \ge M$, we have $x = T^*x = \sigma Tx$ with $\sigma = \frac{M}{\|Tx\|} \le 1$ and $\|x\| = \|T^*x\| = M$. This is a contradiction since (4.10) implies that $\|x\| < M$. Thus $\|Tx\| < M$ and $Tx = x$. □

4.2.1 Application to the existence of solutions to quasi-linear equations

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\partial\Omega$ of class $C^{2,\alpha}$ and $B = C^{1,\beta}(\overline{\Omega})$. Let

$$Qu = a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du)$$

with $a^{ij}(x,z,p) \ge \lambda(x,z,p)\cdot\delta^{ij}$ such that $\lambda(x,z,p) > 0$ for all x,z,p. The boundary values should satisfy $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and $a^{ij}, b \in C^{\alpha}(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$.

Define $T : C^{1,\beta}(\overline{\Omega}) \to C^{2,\beta'}(\overline{\Omega})$ for a $0 < \beta' < \alpha\beta$, such that $u = T(v)$ is the <u>unique</u> solution in $C^{2,\beta'}(\overline{\Omega})$ of the linear Dirichlet problem:

$$\begin{cases} a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}.$$

The unique solvability is guarantueed by Theorem 3.3.5. Furthermore, for $u \in C^{2,\alpha}(\overline{\Omega})$ we have

$$\begin{cases} Qu = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega \end{cases} \iff Tu = u \text{ in } B = C^{1,\beta}(\overline{\Omega})\,,$$

i.e. u is a fixed point of T. We aim to apply Theorem 4.2.7:

$$\begin{aligned} u = \sigma Tu \text{ in } B \iff & \frac{1}{\sigma}u = Tu \\ \iff & \begin{cases} Q_\sigma u = a^{ij}(x,u,Du)D_{ij}u + \sigma b(x,u,Du) = 0 & \text{in } \Omega \\ u = \sigma\varphi & \text{on } \partial\Omega. \end{cases} \end{aligned}$$

Furthermore, note that for any solution of $Tu = \sigma^{-1}u$, since $Tu \in C^{2,\beta'}(\bar{\Omega})$, we have that $u \in C^{2,\beta'}(\bar{\Omega})$, and u solves

$$\begin{cases} \tilde{a}^{ij}(x)D_{ij}u = -\sigma b(x,u,Du) := f(x) & \text{in } \Omega \\ u = \sigma\varphi & \text{on } \partial\Omega. \end{cases}$$

with $\tilde{a}^{ij} := a^{ij}(x,u,Du)$ and $\tilde{a}^{ij}, f \in C^{0,\alpha}(\bar{\Omega})$. The unique solvability of this problem in $C^{2,\alpha}(\bar{\Omega})$, see again Theorem 3.3.5, implies that actually $u \in C^{2,\alpha}(\bar{\Omega})$. Using this we can show:

Theorem 4.2.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let Q be elliptic with $a^{ij}, b \in C^\alpha(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $0 < \alpha < 1$, $\partial\Omega$ of class $C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. If there exists an $0 < \beta \leq \alpha$ and a constant $M > 0$, independent of u and σ, such that every $C^{2,\alpha}(\overline{\Omega})$-solution of $Q_\sigma u = 0$ in Ω with $u = \sigma\varphi$ on $\partial\Omega$, $0 \leq \sigma \leq 1$ fulfills*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} < M,$$

then the Dirichlet problem

$$\begin{cases} Qu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

has a solution in $C^{2,\alpha}(\overline{\Omega})$.

Proof. To apply Theorem 4.2.7 we only have to show that T is compact and continuous.

By the global Schauder estimates T maps bounded sets in $C^{1,\beta}(\overline{\Omega})$ to bounded sets in $C^{2,\beta'}(\overline{\Omega})$. But $C^{2,\beta'}(\overline{\Omega})$ is compactly embedded in $C^2(\overline{\Omega})$ and $C^{1,\beta}(\overline{\Omega})$. Thus T is compact.

To show continuity assmue $v_m \to v$ in $C^{1,\beta}(\overline{\Omega})$. Since $\{Tv_m\}$ is precompact in $C^2(\overline{\Omega})$, there is a subsequence $v_{m'}$ such that

$$Tv_{m'} \to u \text{ in } C^2(\overline{\Omega}).$$

Then

$$\begin{aligned} a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) &= \lim_{m' \to \infty} \Big(a^{ij}(x, v_{m'}, Dv_{m'}) D_{ij}(Tv_{m'}) + b(x, v_{m'}, Dv_{m'}) \Big) \\ &= 0 \end{aligned}$$

Since the solutions to

$$a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) = 0$$

are unique, we have

$$Tv_m \to u = Tv \text{ in } C^2(\overline{\Omega}) \hookrightarrow C^{1,\beta}(\overline{\Omega})\,.$$

This implies that T is continous. □

4.3 Examples of quasi-linear equations

In this section we will discuss examples of equations of the general form

$$Qu = a^{ij}(x,u,Du)D_{ij}u + b(x,u,Du)$$

with $a^{ij}(x,z,p) \geq \lambda(x,z,p) \cdot \delta^{ij}$, for some $\lambda(x,z,p) > 0$.

Examples 4.3.1. *(i) Semi-linear equations:*
$\Delta u = |u|^p$, $p > 1$
$\Delta u = f(x, u)$

(ii) p-harmonic functions: These are critial points of the energy $E(u) = \int_\Omega |Du|^p dx$. *They satisfy the Euler-Lagrange equation:* $\operatorname{div}(|Du|^{p-2}Du) = 0$ *(Note for* $p = 2$ *this is the usual laplace equation* $\Delta u = 0$*).*

(iii) In general: Take as an energy $E(u) = \int_\Omega F(x, u, Du)dx$ *for* $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. *We compute its Euler-Lagrange equation, that is for* $\varphi \in C_c^2(\Omega)$*:*

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\Big|_{t=0} E(u + t\varphi) \\
&= \int_\Omega \frac{\partial}{\partial t}\Big|_{t=0} F(x, u + t\varphi, Du + tD\varphi)\, dx \\
&= \int_\Omega \frac{\partial F}{\partial z}(x, u, Du) \cdot \varphi + \frac{\partial F}{\partial p_i}(x, u, Du) \cdot D_i\varphi \ dx \\
&= \int_\Omega \Big(-\operatorname{div}\big(\frac{\partial F}{\partial p_i}(x, u, Du)\big) + \frac{\partial F}{\partial z}(x, u, Du)\Big) \cdot \varphi \ dx \qquad \forall\, \varphi \in C_c^2(\Omega)\,.
\end{aligned}$$

That implies

$$\begin{aligned}
0 &= D_i\left(\frac{\partial F}{\partial p_i}(x, u, Du)\right) - \frac{\partial F}{\partial z}(x, u, Du) \\
\Leftrightarrow 0 &= \underbrace{\frac{\partial^2 F}{\partial p_i \partial p_j}(x, u, Du)}_{=a^{ij}(x,u,Du)} D_iD_ju \\
&\quad + \underbrace{\frac{\partial^2 F}{\partial p_i \partial z}(x, u, Du)D_iu + \sum_i \frac{\partial^2 F}{\partial x_i \partial p_i}(x, u, Du) - \frac{\partial F}{\partial z}(x, u, Du)}_{=b(x,u,Du)},
\end{aligned}$$

that is, a^{ij} *ist elliptic, iff* F *is (strictly) loally convex in the p-variable. For p-harmonic funktions, i.e. we have* $F(x, u, Du) = |Du|^p$ *is convex for* $p \geq 1$. *We can check*

$$\begin{aligned}
\operatorname{div}\left(|Du|^{p-2}Du\right) &= |Du|^{p-2} \cdot \Delta u + (p-2)|Du|^{p-4}D^iuD^juD_{ij}u \\
&= \left(|Du|^{p-2}\delta^{ij} + (p-2)|Du|^{p-24}D^iuD^ju\right) D_{ij}u
\end{aligned}$$

and

$$\begin{aligned}
a^{ij}\,\xi_i\xi_j &= \left(|Du|^{p-2}\delta_{ij} + (p-2)|Du|^{p-4}D_iuD_ju\right)\xi_i\xi_j \\
&= |Du|^{p-2}|\xi|^2 + (p-2)|Du|^{p-4} < Du, \xi >^2 \geq \min\{p-1, 1\} \cdot |Du|^{p-2}|\xi|^2\,.
\end{aligned}$$

(iv) Capillary surfaces: Let $\Omega \subset \mathbb{R}^n$ be a domain. We want to look for critial points of

$$E(u) = \underbrace{\int_\Omega \sqrt{1+|Du|^2}\,dx}_{\text{surface energy}} + \underbrace{\frac{\kappa}{2}\int_\Omega u^2\,dx}_{\text{potential energy}} + \underbrace{\int_{\partial\Omega} \beta u\,d\mathcal{H}^n}_{\text{wetting energy}}$$

with $\kappa \geq 0$, $|\beta| \leq 1-a$ for a $0 < a \leq 1$. Note that β defines the contact angle (see exercises). Critial points of E satisfy

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \kappa u \text{ in } \Omega$$
$$< \nu, \gamma >= \beta \text{ in } \partial\Omega$$

where ν is the is the upper unit normal to the surface, and γ the outward unit normal to $\partial\Omega$.

4.3.1 Future strategy

We have so far: If we have a-priori estimates of the form

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$$

then we can show that a solution in $C^{2,\alpha}(\overline{\Omega})$ exists.

Strategy to obtain a-priori estimates in $C^{1,\alpha}$:

(i) Estimate

$$\sup_\Omega |u|\,.$$

This can often be done by constructing global upper and lower barriers.

(ii) Estimate

$$\sup_{\partial\Omega} |Du|\,.$$

This can be done by constructing barriers at the boundary. This often uses (i) and sometimes gives restrictions on the geometry of the boundary $\partial\Omega$.

(iii) Estimate

$$\sup_\Omega |Du| \leq C(\sup_{\partial\Omega} |Du|, \sup_\Omega |u|, \Omega)\,.$$

This can be done by proving a maximum principle for the gradients, which works under strucutral conditions on $a^{ij}(x,z,p)$ and $b(x,z,p)$.

(iv) Estimate

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C(\sup_\Omega |Du|, \sup_\Omega |u|, \Omega)\,.$$

Here the DeGiorgi-Nash-Moser estimates are needed.

4.3.2 The first three steps for minimal surfaces

If graph(u) is a minimal surface with boundary values φ then u satisfies

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{4.11}$$

i) Observe that the graph of any linear function is a minimal surface. By the weak maximum principle this implies

$$\sup_\Omega u \le \sup_{\partial\Omega} u = \sup_{\partial\Omega} \varphi$$
$$\inf_\Omega u \ge \inf_{\partial\Omega} u = \inf_{\partial\Omega} \varphi$$

ii) Let us assume that we have the following simpler case: We assume that the boundary values $(\partial\Omega, \varphi)$ satisfy an upper and lower slope condition, that is for all $x_0 \in \partial\Omega$ there exist hyperplanes $P^+_{x_0}$, $P^-_{x_0}$ in $\mathbb{R}^{n+1}$ with $|\nabla P^\pm_{x_0}| \le C$, C independent of x_0, such that

$$P^-_{x_0} \le \operatorname{graph} \varphi \le P^+_{x_0} \qquad \text{and}$$
$$P^-_{x_0}(x_0) = \varphi(x_0) = P^+_{x_0}(x_0).$$

Since hyperplanes are minimal surfaces we have by the weak maximum principle that

$$P^-_{x_0}(x) \le u(x) \le P^+_{x_0}(x) \quad \forall x \in \Omega$$

which implies

$$\left|\frac{\partial u}{\partial \nu}(x_0)\right| \le C$$

and thus

$$|Du|(x_0) \le C\left|\frac{\partial u}{\partial \nu}(x_0)\right| + C|D^T\varphi(x_0)| \le C'$$

iii) By differentiating (4.11), see Lemma 4.1.4, we have seen that every partial derivative $w = D_l u$ is a solution of

$$D_i(a^{ij}(Du)D_j w) = 0 \qquad \text{on } \Omega\,.$$

The weak maximum principle then implies

$$\sup_\Omega |w| \le \sup_{\partial\Omega} |w|\,,$$

which yields

$$\sup_\Omega |Du| \le C \sup_{\partial\Omega} |Du|.$$

5 DeGiorgi-Nash-Moser estimates

5.1 A weak version of the Harnack inequality

We first collect some analysis facts which we will need in the sequel:

i) Let $\Omega \subset \mathbb{R}$ be a bounded domain, and $u : \Omega \to \mathbb{R}$ mesurable. Then it holds that

$$\lim_{p \to \infty} \left(\frac{1}{|\Omega|} \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} = \operatorname{ess\,sup}_\Omega |u|$$

and

$$\lim_{p \to -\infty} \left(\frac{1}{|\Omega|} \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} = \operatorname{ess\,inf}_\Omega |u| \ .$$

ii) Sobolev inequality: Let $u \in W_0^{1,p}(\Omega)$, then it holds that

$$\left(\int_\Omega |u|^{\frac{np}{n-p}} \right)^{\frac{n-p}{np}} \leq C(n,p) \left(\int_\Omega |Du|^p \right)^{\frac{1}{p}}, \qquad \text{for } 1 \leq p < n$$

and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}$ is continuous. Furthermore the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega), \qquad 1 \leq r \leq \frac{np}{n-p} =: p^*$$

is compact (Rellich-Kondrachov).

iii) Poincaré inequality: Let $u \in W_0^{1,p}(\Omega)$, for $p < n$. Then we have

$$\left(\int_\Omega |u|^p \right)^{\frac{1}{p}} \leq C(n,p) |\Omega|^{\frac{1}{n}} \left(\int_\Omega |Du|^p \right)^{\frac{1}{p}} .$$

If the dimension is $n = 2$, more is true. Consider the Sobolev inequality in this case for $p = 1$:

$$\left(\int_\Omega |u|^2 \right)^{\frac{1}{2}} \leq C \int_\Omega |Du|.$$

We write $u = v^r$ and obtain

$$\left(\int_\Omega |v|^{2r}\right)^{\frac{1}{2}} \leq C\int_\Omega r|v|^{r-1}|Dv|$$
$$\leq Cr\left(\int_\Omega |Dv|^2\right)^{\frac{1}{2}}\left(\int_\Omega |v|^{2r-2}\right)^{\frac{1}{2}}$$
$$\leq Cr\left(\int_\Omega |Dv|^2\right)^{\frac{1}{2}}\left(\int_\Omega |v|^{2r}\right)^{\frac{r-1}{2r}}|\Omega|^{\frac{1}{2r}},$$

which yields

$$\left(\int_\Omega |v|^{2r}\right)^{\frac{1}{2r}} \leq Cr|\Omega|^{\frac{1}{2r}}\left(\int_\Omega |Dv|^2\right)^{\frac{1}{2}},$$

that is, we obtain a Poincaré inquality for any $1 \leq p < \infty$!

In this chapter we will investigate solutions to elliptic equations in divergence form, that is solutions u to equations of the type

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^iD_iu + du = D^if_i + g \tag{5.1}$$

with $a^{ij} \geq \lambda \cdot \delta^{ij}$, for some $\lambda > 0$. Multiplying (5.1) by $2/\lambda$, and replacing $2a^{ij}/\lambda \to a^{ij}$, $2b^i/\lambda \to b^i$, $2f^i/\lambda \to f^i$ we can assume that

$$a^{ij} \geq 2\delta^{ij}.$$

To simplify our computations later, we rewrite (5.1) in the form

$$D_iA^i(x,u,Du) + B(x,u,Du) = 0 \tag{5.2}$$

with

$$A^i(x,z,p) = a^{ij}p_j + b^iz - f^i$$
$$B(x,z,p) = c^ip_i + dz - g.$$

We further assume that

$$\sum_{i,j}|a^{ij}|^2 \leq \Lambda^2, \qquad \sum_i |b^i|^2 + |c^i|^2 + |d| \leq \nu^2 \tag{5.3}$$

for some constants Λ, $\nu > 0$. We furthermore introduce the quantities

$$\bar{z} := |z| + k, \quad \bar{b} := \left(|b|^2 + |c|^2 + \frac{|f|^2}{k^2}\right) + \left(|d| + \frac{|g|}{k}\right)$$

for some $k > 0$.

Proposition 5.1.1. *The following estimates hold*

i) $|A(x,z,p)| \le |a|\,|p| + 2(\bar b)^{1/2}\bar z$

ii) $p^i A^i(x,z,p) \ge |p|^2 - 2\bar b \bar z^2$

iii) $|\bar z B(x,z,p)| \le \varepsilon |p|^2 + \frac{1}{\varepsilon}\bar b \bar z^2$ *for every* $0 < \varepsilon \le 1$.

Proof. *i)* We can directly estimate

$$\begin{aligned} A(x,z,p) &\le |a|\,|p| + |b|\,|z| + |f| \\ &\le |a|\,|p| + |b|\,\bar z + \frac{|f|}{k}\bar z \\ &\le |a|\,|p| + 2(\bar b)^{1/2}\bar z\,. \end{aligned}$$

ii) Using Young's inequality we have

$$\begin{aligned} p^i A^i(x,z,p) &\ge 2|p|^2 - |p|\,|b|\,|z| - |p|\,|f| \\ &\ge |p|^2 - \frac{1}{2}|b|\,|z|^2 - \frac{1}{2}|f|^2 \\ &\ge |p|^2 - \frac{1}{2}|b|\,\bar z^2 - \frac{1}{2}\frac{|f|^2}{k^2}\bar z^2 \\ &\ge |p|^2 - 2\bar b \bar z^2\,. \end{aligned}$$

iii) Similarly

$$\begin{aligned} |\bar z B(x,z,p)| &\le |\bar z|\,|c|\,|p| + \bar z\,|d|\,|z| + \bar z\,|g| \\ &\le \varepsilon |p|^2 + \frac{1}{4\varepsilon}\bar z^2 |c|^2 + \bar z^2 |d| + \frac{|g|}{k}\bar z^2 \\ &\le \varepsilon |p|^2 + \frac{1}{\varepsilon}|\bar b|\,\bar z^2\,. \end{aligned}$$

□

We define

$$K(R) := K = \frac{1}{\lambda}\left(R^{1-\frac{n}{q}}\|f\|_{L^q} + R^{2\left(1-\frac{q}{n}\right)}\|g\|_{L^{q/2}}\right)$$

where in the rescaled setting we have $\lambda = 2$. In this section we will prove the following to estimates:

Theorem 5.1.2. *Let* L *be an operator as in* (5.1), *satisfying* (5.3), *with* $a^{ij} \ge \lambda \delta^{ji}$ *and* $f^i \in L^q(\Omega)$, $i = 1, \dots, n$, $g \in L^{q/2}(\Omega)$ *for a* $q > n$. *If* $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ *is a subsolution (i.e.* $Lu \ge D_i f^i + g$*) of* (5.1), *which is non-negative on* $B_{4R}(y) \subset \Omega$, *then*

$$\sup_{B_R(y)} u \le C\left(R^{-\frac{n}{p}}\|u\|_{L^p(B_{2R}(y))} + K(R)\right)$$

for $p > 1$ *and* $C = C(n, 1/\lambda, \nu R, q, p)$.

Theorem 5.1.3. *Let L be as in Theorem 5.1.2. If $u \in W^{1,2}(\Omega)$ is a supersolution (i.e. $Lu \leq D_i f^i + g$) of (5.1), which is non-negative on $B_{4R}(y) \subset \Omega$ and $1 \leq p < n/n-2$, then*

$$R^{-\frac{n}{p}} \|u\|_{L^p(B_{2R}(y))} \leq C \left(\inf_{B_R(y)} u + K(R) \right)$$

with $C = C(n, 1/\lambda, \nu R, q, p)$.

Remark 5.1.4: Theorem 5.1.3 can be seen as a weak Harnack inequality.

Preparation for the proof of Theorem 5.1.2 and 5.1.3 together, following [3]: By definition, if u is a weak subsolution, (resp. supersolution), of (5.2), then

$$\int_\Omega \left(D_i v A^i(x,u,Du) - vB(x,u,Du) \right) dx \leq 0 \quad (\geq 0) \qquad \forall\, v \geq 0, v \in W_0^{1,2}(\Omega)\,. \tag{5.4}$$

We write $B_R = B_R(y)$ and assume w.l.o.g. that $R = 1$ and $k > 0$. The general case follows by the transformation $x \mapsto x/R$, and $k \to 0$. Let

$$v := \eta^2 \bar{u}^\beta$$

for $\bar{u} := u + k$, $\beta \neq 0$, $\eta \in C_0^1(B_4)$, $\eta \geq 0$. We want to use v as a test function in (5.4). Note that we have assumed that $u \in L^\infty$ so $v \in W^{1,2}(\Omega)$. We have

$$Dv = 2\eta D\eta \bar{u}^\beta + \beta \eta^2 \bar{u}^{\beta-1} Du$$

and v is a valid test function in (5.4). Inserting this into (5.4) yields

$$\beta \int_\Omega \eta^2 \bar{u}^{\beta-1} D_i u A^i(x,u,Du)\, dx + 2 \int_\Omega \eta D_i \eta A^i(x,u,Du) \bar{u}^\beta\, dx \\ - \int_\Omega \eta^2 \bar{u}^\beta B(x,u,Du)\, dx \leq 0$$

respectively "≥ 0" in the case of a supersolution. The estimates from Proposition 5.1.1 yield

$$\eta^2 \bar{u}^{\beta-1} D_i u A^i(x,u,Du) \geq \eta^2 \bar{u}^{\beta-1} |Du|^2 - 2\bar{b} \cdot \eta^2 \bar{u}^{\beta+1}$$

$$\begin{aligned} |\eta D_i \eta A^i(x,u,Du) \bar{u}^\beta| &\leq |a| \eta |D\eta| \bar{u}^\beta |Du| + 2|\bar{b}|^{1/2} \eta |D\eta| \bar{u}^{\beta+1} \\ &\leq \frac{\epsilon}{2} \eta^2 \bar{u}^{\beta-1} |Du|^2 + \left(1 + \frac{|a|^2}{2\epsilon}\right) |D\eta|^2 \bar{u}^{\beta+1} + \bar{b} \eta^2 \bar{u}^{\beta+1} \end{aligned}$$

$$|\eta^2 \bar{u}^\beta B(x,u,Du)| \leq \epsilon \eta^2 \bar{u}^{\beta-1} |Du|^2 + \frac{1}{\epsilon} \bar{b} \eta^2 \bar{u}^{\beta+1}$$

In case of a subsolution, assuming that $\beta > 0$, and inserting this into the estimate before we obtain

$$\beta \int_\Omega \eta^2 \bar{u}^{\beta-1} |Du|^2 \, dx \le \int_\Omega 2\epsilon \eta^2 \bar{u}^{\beta-1} |Du|^2 \, dx + \int_\Omega (2\beta + 2 + \frac{1}{\epsilon}) \bar{b} \eta^2 \bar{u}^{\beta+1} + \left(2 + \frac{|a|^2}{\epsilon}\right) |D\eta|^2 \bar{u}^{\beta+1} \, dx \, .$$

For supersolution we obtain the reverse inequality with a minus sign in front of the right hand side. We now always assume, that if u is a subsolution, then we choose $\beta > 0$ and if u is a supersolution we choose $\beta < 0$. We choose $\epsilon := \min\{1, \frac{|\beta|}{4}\}$, bring the $|Du|^2$-term to the left hand side, and divide by $\beta/2$. This yields in the case of a super- and a subsolution:

$$\int_\Omega \eta^2 \bar{u}^{\beta-1} |Du|^2 \, dx \le C(|\beta|) \int_\Omega (\bar{b}\eta^2 + (1 + |a|^2) |D\eta|^2) \bar{u}^{\beta+1} \, dx$$

where $C(|\beta|)$ explodes if $|\beta| \to 0$. So we will always assume in the following that $|\beta| \ge \delta > 0$. We take

$$w := \begin{cases} \bar{u}^{\frac{\beta+1}{2}} & \text{if } \beta \ne -1 \\ \log \bar{u} & \text{if } \beta = -1 \end{cases}$$

for $\gamma := \beta + 1$ and obtain

$$\int_\Omega |\eta D w|^2 \, dx \le \begin{cases} C(|\beta|)\gamma^2 \int_\Omega (\bar{b}\eta^2 + (1 + |a|^2)|D\eta|^2) w^2 \, dx & \text{for } \beta \ne -1 \\ C \int_\Omega (\bar{b}\eta^2 + (1 + |a|^2)|D\eta|^2) \, dx & \text{for } \beta = -1 \end{cases} \tag{5.5}$$

We first assume that $\beta \ne -1$. By the Sobolev inequality we see

$$\|\eta w\|^2_{\frac{2\widehat{n}}{\widehat{n}-2}} \le C(\widehat{n}) \int_\Omega |\eta D w|^2 + |w D\eta|^2 \, dx$$

$$\widehat{n} = n \text{ for } n > 2$$

$$2 < \widehat{n} < q \text{ for } n = 2$$

and by the Hölder-inequality and the interpolation of L^p-norms

$$\begin{aligned} \int_\Omega \bar{b} (\eta w)^2 dx &\le \|\bar{b}\|_{\frac{q}{2}, B_4} \cdot \|\eta w\|^2_{\frac{2q}{q-2}} \\ &\le \|\bar{b}\|_{\frac{q}{2}, B_4} \cdot (\epsilon \|\eta w\|_{\frac{2\widehat{n}}{\widehat{n}-2}} + \epsilon^{-\sigma} \|\eta w\|_2)^2, \end{aligned} \tag{5.6}$$

where $\sigma = \frac{\widehat{n}}{q - \widehat{n}}$. Note that $q > \widehat{n}$ by definition. Recall that

$$\bar{b} = \left(|b|^2 + |c|^2 + \frac{|f|^2}{k^2}\right) + \left(|d| + \frac{|g|}{k}\right)$$

and thus

$$\|\bar{b}\|_{\frac{q}{2},B_4} \leq C(\nu) + \frac{2}{k^2}\left(\int_{B_4} |f|^q\right)^{\frac{2}{q}} + \frac{2}{k}\left(\int_{B_4} |g|^{\frac{q}{2}}\right)^{\frac{2}{q}} \leq C(\nu) + 4$$

provided

$$k \geq k_0 = \|f\|_{q,B_4} + \|g\|_{\frac{q}{2},B_4}$$

Putting this together we have

$$\begin{aligned}\|\eta w\|_{\frac{2\widehat{n}}{\widehat{n}-2}} &\leq \gamma^2 C'(|\beta|)\int_\Omega \bar{b}(\eta w)^2\,dx + C(|\beta|)(1+\gamma^2)\int_\Omega (1+\Lambda^2)|D\eta|^2 w^2\,dx \\ &\leq \gamma^2 C''(|\beta|,\nu)(\epsilon^2\|\eta w\|^2_{\frac{2\widehat{n}}{\widehat{n}-2}} + \epsilon^{-2\sigma}\|\eta w\|_2^2) + C(|\beta|,\Lambda)(1+\gamma^2)\int_\Omega |D\eta|^2 w^2\,dx\end{aligned}$$

We choose

$$\epsilon = \frac{1}{C''(|\beta|,\nu)^{\frac{1}{2}}}\cdot\frac{1}{\sqrt{2}|\gamma|}$$

and arrive at

$$\|\eta w\|_{\frac{2\widehat{n}}{\widehat{n}-2}} \leq C(1+|\gamma|)^{\sigma+1}\|(\eta+|D\eta|)w\|_2$$

with $C = C(\widehat{n},\Lambda,\nu,q,|\beta|^{-1})$. Let $1 \leq r_1 \leq r_2 \leq 3$ and choose a cut-off function such that

$$\begin{aligned}\eta &\equiv 1 \text{ on } B_{r_1}\\ \eta &\equiv 0 \text{ on } \Omega\setminus B_{r_2}\\ |D\eta| &\leq \frac{2}{r_2-r_1}\end{aligned}$$

Define $\chi := \frac{\widehat{n}}{\widehat{n}-2} > 1$. We thus see that

$$\|w\|_{L^{2\chi}(B_{r_1})} \leq \frac{C(1+|\gamma|)^{\sigma+1}}{r_2-r_1}\|w\|_{L^2(B_{r_2})} \tag{5.7}$$

The idea is now to iterate (5.7) to obtain a sup-estimate.

We define for $r < 4$, $p \neq 0$

$$\Phi(p,r) = \left(\int_{B_r} \bar{u}^p dx\right)^{\frac{1}{p}}$$

and recall that

$$\lim_{p\to+\infty}\Phi(p,r) = \sup_{B_r}\bar{u}, \qquad \lim_{p\to-\infty}\Phi(p,r) = \inf_{B_r}\bar{u}\ .$$

We obtain from (5.7), $\gamma = \beta + 1$, $1 \le r_1 < r_2 \le 3$, that

$$\Phi(\chi\gamma, r_1) \le \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1}\right)^{\frac{2}{|\gamma|}} \Phi(\gamma, r_2), \qquad \text{if } \gamma > 0 \tag{5.8}$$

Proof of Theorem 5.1.2. We now assume that we have a supersolution, that is $\beta > 0$, $\gamma > 1$. We choose $p > 1$ arbitrary, and set $\gamma = \gamma_m = \chi^m p$ iteratively and $r_m := 1 + 2^{-m}$, for $m = 0, 1, 2, \ldots$. This yields

$$\begin{aligned}
\Phi(\chi^{m+1}p, 1) &\le \left(\frac{C(1+\chi^m p)^{\sigma+1}}{2^{-m}}\right)^{\frac{2}{p}\chi^{-m}} \Phi(\chi^m p, 1 + 2^{-m}) \\
&\le \left(\frac{C(1+\chi^m p)^{\sigma+1}}{2^{-m}}\right)^{\frac{2}{p}\chi^{-m}} \cdot \Big(\ldots\Big)(\ldots)(\ldots) \cdot \Phi(p, 2) \\
&\le (C\cdot\chi)^{2mp^{-1}(\sigma+1)\chi^{-m}} \cdot \Big(\ldots\Big)(\ldots)(\ldots) \cdot \Phi(p, 2) \\
&\le (C\cdot\chi)^{2(\sigma+1)\sum_m m\chi^{-m}} \cdot \Phi(p, 2) \\
&\le C(\widehat{n}, \Lambda, \nu, p, q)\Phi(p, 2)
\end{aligned}$$

and thus for $m \to \infty$ we see that

$$\sup_{B_1} \bar{u} \le C\|\bar{u}\|_{L^p(B_2)}$$

We now choose $k = k_0 = \|f\|_{q,B_4} + \|g\|_{\frac{q}{2},B_4}$. This yields

$$\begin{aligned}
\sup_{B_1} u \le \sup_{B_1} |u + k| \\
&\le C\|u + k\|_{L^p(B_2)} \\
&\le C(\|u\|_{L^p(B_2)} + \|k\|_{L^p(B_2)}) \\
&\le C'(\|u\|_{L^p(B_2)} + k_0)
\end{aligned}$$

This finishes the proof of Theorem 5.1.2. □

Remark 5.1.5: *By varying the testfunction used for the proof of Theorem 5.1.2 one can drop the assumption that u is non-negative and bounded. The statement then reads that if u is a $W^{1,2}$ subsolution (supersolution) of* (5.1) *in Ω, we have, for any Ball $B_{4R}(y) \subset \Omega$ and $p > 1$,*

$$\sup_{B_R(y)} u\,(-u) \le C(R^{-n/p}\|u^+\,(u^-)\|_{L^p(B_{2R}(y))} + K(R))$$

where $C = C(n, \Lambda/\lambda, \nu R, q, p)$. For a sketch of a proof of this statement see [3].

Proof of Theorem 5.1.3. **Reminder:** Wir had introduced in the previous section the notation

$$w = \begin{cases} \bar{u}^{\frac{\beta+1}{2}} & \text{for } \beta \neq -1 \\ \log \bar{u} & \text{for} \beta = -1\,, \end{cases}$$

with $\gamma = \beta + 1$. We considered the case $\beta > 0$, that is that u is a subsolution (5.1). Now we consider the case that u is a supersolution and that $\beta < 0$. Note that in this case we don't have to assume that $u \in L^\infty$ since $\bar{u} \geq k > 0$ and so $w \in W^{1,2}(\Omega)$. We recall that

$$\Phi(p,r) := \left(\int\limits_{B_r} |\bar{u}|^p dx \right)^{\frac{1}{p}}.$$

We aim to prove Theorem 5.1.3, that is we want to show that

$$R^{-\frac{n}{p}} \|u\|_{L^p\left(B_{2R}(y)\right)} \leq c \cdot \left(\inf_{B_R(y)} u + K(R) \right).$$

The assumption $\beta < 0$ implies $\gamma = 1 + \beta < 1$. We have already shown that, see (5.7), that for $1 \leq r_1 < r_2 \leq 3$ we have for $\chi = \frac{\hat{n}}{\hat{n}-2}$

$$\Phi(\chi\gamma, r_1) \leq \left(\frac{C \cdot (1 + |\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{\frac{2}{|\gamma|}} \Phi(\gamma, r_2) \tag{5.9}$$

if $0 < \gamma < 1$, and

$$\Phi(\gamma, r_2) \leq \left(\frac{C \cdot (1 + |\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{\frac{2}{|\gamma|}} \Phi(\chi\gamma, r_1) \tag{5.10}$$

for $\gamma < 0$, where C is bounded, as long as $|\gamma|$ is bounded away from zero. Assume that $0 < p_0 < p < \chi$, then we get from the iteration in the proof of Theorem 5.1.2:

$$\Phi(p, 2) \leq C\Phi(p_0, 3).$$

Attention: We need $p < \chi$, since with $\gamma < 1$ we have that $\chi\gamma < \chi$. So we pick γ such that $\chi\gamma = p$, that is $\gamma = \frac{p}{\chi}$. We then iterate backwards until $\frac{p}{\chi^m} \leq p_0$.

Starting with $-p_0$ we can iterate (5.10) as in the previous proof to obtain:

$$\Phi(-p_0, 3) \leq C(\hat{n}, \Lambda, \nu, p, q, p_0)\Phi(-\infty, 1).$$

To complete the proof we need the following lemma:

Lemma 5.1.6. *There exists $p_0 > 0$ with $0 < p_0 < \chi = \frac{\widehat{n}}{\widehat{n}-2}$ such that*

$$\Phi(p_0, 3) \leq C \cdot \Phi(-p_0, 3)\,,$$

where $C = C(\hat{n}, \gamma, \Lambda)$.

This yields for $k = k_0 = \|f\|_{q,B_4} + \|g\|_{q/2,B_4}$

$$\|u\|_{p,B_2} \leq \|u + k_0\|_{p,B_2} \leq C \cdot \inf_{B_1} \bar{u} = C \cdot \left(\inf_{B_1} u + \|f\|_{q,B_4} + \|g\|_{q/2,B_4} \right)$$

which is the statement of Theorem 5.1.3 □

Proof of Lemma 5.1.6. We now apply the estimate (5.5) for $\beta = -1$, that is $\gamma = 0$. By choosing the aprropriate cut-off function η, more precisely for $0 < r < 1$

$$\eta = \begin{cases} 1 & \text{on } B_r(x) \\ 0 & \text{on } \Omega \setminus B_{2r}(x) \end{cases}$$

with $|D\eta| \leq \frac{2}{r}$, we have

$$\begin{aligned} \Big(\int_{B_r(x)} |Dw|^2 \Big)^{\frac{1}{2}} &\leq C \Big(\int_{\Omega} \bar{b}\eta^2 + (1 + |a|^2)|D\eta|^2 dx \Big)^{\frac{1}{2}} \\ &\leq C \Big(\int_{B_{2r}(x)} \bar{b} + (1 + \Lambda^2)\frac{1}{r^2} dx \Big)^{\frac{1}{2}} \\ &\leq C \Big(\|\bar{b}\|_{L^{\frac{q}{2}}} (2r)^{n\frac{q-2}{q}} + (1 + \Lambda^2) 2^n r^{n-2} \Big)^{\frac{1}{2}} \\ &\leq C (r^{n-2})^{\frac{1}{2}} \\ &= C r^{\frac{n}{2} - 1}, \end{aligned}$$

since for $q > n$ we have $\frac{n}{q}(q - 2) > n - 2$. This implies

$$\int_{B_r(x)} |Dw| dx \leq C r^{\frac{n}{2}} \left(\int_{B_r(x)} |Dw|^2 dx \right)^{\frac{1}{2}} \leq C r^{n-1}.$$

The John-Nirenberg estimate, which we will prove in the following section, see 5.2.6, implies that there is a constant $p_0 > 0$, (w.l.o.g $p_0 < \chi$), with $p_0 = p_0(n, \Lambda, \nu)$, such that

$$\int_{B_3} \exp(p_0 |w - w_0|) \leq C(n, \Lambda, \nu)$$

with $w_0 = \frac{1}{|B_3|} \int_{B_3} w dx$. This implies

$$\begin{aligned} \int_{B_3} \exp(p_0 w)\, dx \cdot \int_{B_3} \exp(-p_0 w)\, dx &= \int_{B_3} \exp(p_0 w - p_0 w_0)\, dx \cdot \int_{B_3} \exp(p_0 w_0 - p_0 w)\, dx \\ &\leq C. \end{aligned}$$

With $w = \log \bar{u}$ this implies

$$\left(\int\limits_{B_3} \bar{u}^{p_0} dx\right)^{\frac{1}{p_0}} \leq C^{\frac{1}{p_0}} \left(\int\limits_{B_3} \bar{u}^{-p_0}\right)^{-\frac{1}{p_0}}$$

□

Remark 5.1.7: The John-Nirenberg estimate is a limiting case of the Sobolev inequality. Let $u \in W_0^{1,p}(B_r(0))$ with $p < n$, then by Sobolev

$$\left(\int |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \leq C \left(\int |Du|^p\right)^{\frac{1}{p}} .$$

For $p \to n$ this is not anymore valid, but we have for $p = n$ that

$$\left(\int |Du| dx\right) \leq C \left(\int |Du|^n dx\right)^{\frac{1}{n}} (r^n)^{1-\frac{1}{n}} \leq C r^{n-1} .$$

The John-Nirenberg estimate now implies by writing out the exponential series that

$$\int 1 + p_0(|u - u_0|) + \frac{1}{2}(p_0(|u - u_0|))^2 + \cdots \leq C.$$

The Sobolev inequality already implied that $u \in L^p(B_r(0))$ for all $p \geq 1$, but not the embedding into L^∞. We don't achieve this here either, but a uniform bound for an appropriate series of L^p-norms.

5.2 A short digression to potential estimates

Let $\mu \in (0, 1]$ and define the operator V_μ on $L^1(\Omega)$ by

$$V_\mu(f)(x) = \int\limits_\Omega |x - y|^{n(\mu-1)} f(y) dy,$$

called the Riesz potential. That V_μ is well defined and maps L^1 into L^1 will follow by the next lemma. First note that

$$\begin{aligned} V_\mu(1)(x) &= \int\limits_\Omega |x - y|^{n(\mu-1)} dy \\ &\leq \int\limits_{B_R(x)} |x - y|^{n(\mu-1)} d\mu \\ &= \frac{1}{\mu} \omega_n R^{n\mu} \\ &= \mu^{-1} \omega_n^{1-\mu} |\Omega|^\mu, \end{aligned}$$

where we have chosen R such that $|\Omega| = |B_R(x)| = \omega_n R^n$.

Lemma 5.2.1. *The operator V_μ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ for any q with $1 \le q \le \infty$ satisfying*

$$0 \le \delta := \frac{1}{p} - \frac{1}{q} < \mu$$

Furthermore for any $f \in L^p(\Omega)$

$$\|V_\mu f\|_{L^q(\Omega)} \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_{L^p(\Omega)}.$$

Proof. Choose $r \ge 1$ s.t.

$$\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \delta\ .$$

Note that this implies $1 = (1 - \frac{1}{p}) + \frac{1}{q} + \delta$. Then define $h(x-y) := |x-y|^{n(\mu-1)}$. This implies

$$\begin{aligned}\big(h(x-y)\big)^r &= \big(|x-y|^{n(\mu-1)}\big)^r \\ &= |x-y|^{n\frac{\mu-1}{1-\delta}} \\ &= |x-y|^{n\left(\frac{\mu-\delta}{1-\delta}-1\right)}\end{aligned}$$

and

$$\frac{\mu-\delta}{1-\delta} \in (0,1]\,.$$

Furthermore,

$$\begin{aligned}\left(\int_\Omega h^r(x-y)\right)^{\frac{1}{r}} &\le \left(\frac{1-\delta}{\mu-\delta}\right)^{\frac{1}{r}} \omega_n^{\frac{1}{r}\left(1-\frac{\mu-\delta}{1-\delta}\right)} |\Omega|^{\frac{1}{r}\left(\frac{\mu-\delta}{1-\delta}\right)} \\ &= \left(\frac{1-\delta}{\mu-\delta}\right)^{\frac{1}{r}} \omega_n^{1-\mu} |\Omega|^{\mu-\delta}\,.\end{aligned}$$

We write

$$h \cdot |f| = h^{\frac{r}{q}} h^{r\left(1-\frac{1}{p}\right)} |f|^{\frac{p}{q}} |f|^{p\delta}$$

and thus by Hölder:

$$\begin{aligned}|V_\mu f(x)| &\le \int_\Omega h|f|dy \\ &\le \left(\int_\Omega h^r(x-y)|f(y)|^p dy\right)^{\frac{1}{q}} \left(\int_\Omega h^r(x-y)dy\right)^{1-\frac{1}{p}} \\ &\quad \cdot \left(\int_\Omega |f(y)|^p dy\right)^{\delta}\end{aligned}$$

Thus we have

$$
\begin{aligned}
\|V_\mu f\|^q_{L^q(\Omega)} &\le \|f\|^{\delta pq}_{L^p(\Omega)} \cdot \sup_{x\in\Omega}\left(\int\limits_\Omega h^r(x-y)dy\right)^{q-\frac{q}{p}} \cdot \int\limits_\Omega\int\limits_\Omega h^r(x-y)|f(y)|^p dydx \\
&\le \|f\|^{\delta pq}_{L^p(\Omega)} \cdot \sup_{x\in\Omega}\left(\int\limits_\Omega h^r(x-y)dy\right)^{q-\frac{q}{p}+1} \|f\|^p_{L^p(\Omega)} \\
&\le \|f\|^{(1+\delta q)p}_{L^p(\Omega)} \cdot \sup_{x\in\Omega}\left(\int\limits_\Omega h^r(x-y)dy\right)^{\frac{q}{r}}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|V_\mu f\|_{L^q(\Omega)} &\le \|f\|^{\frac{p}{q}+\delta p}_{L^p(\Omega)} \cdot \sup_{x\in\Omega}\left(\int\limits_\Omega h^r(x-y)dy\right)^{\frac{1}{r}} \\
&\le \|f\|_{L^p(\Omega)}\left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu}|\Omega|^{\mu-\delta}\,.
\end{aligned}
$$

□

In the following denote with u_Ω the mean value of u on Ω. We denote with d the diameter of the set in question, which is here $\operatorname{diam}\Omega$.

Lemma 5.2.2. *Let Ω be convex and $u \in W^{1,1}(\Omega)$ and $S \subset \Omega$ be measurable with $|S| > 0$. Then*

$$
|u(x) - u_S| \le \frac{d^n}{n|S|}\int\limits_\Omega |x-y|^{1-n}|Du(y)|dy \quad \textit{for a.e. } x\in\Omega\,. \tag{5.11}
$$

Proof. Since $C^1(\Omega)$ is dense in $W^{1,1}(\Omega)$, it suffices to to prove (5.11) for C^1-functions. Thus for $x, y \in \Omega$:

$$
u(x) - u(y) = -\int\limits_0^{|x-y|} D_r u(x + r\omega)dr, \text{ where } \omega = \frac{y-x}{|x-y|}\,.
$$

Integrating w.r.t y over S gives

$$
|S|(u(x) - u_S) = -\int\limits_S (\int\limits_0^{|x-y|} D_r u(x + r\omega)dr)dy\,.
$$

Extending $D_r(u(x+r\omega))$ by 0 outside Ω we obtain:

$$\begin{aligned}
|u(x)-u_S| &\le \frac{1}{|S|}\int\limits_{B_d(x)}\int\limits_0^\infty |D_r u(x+r\omega)|drdy \\
&\le \frac{1}{|S|}\int\limits_0^\infty\int\limits_{|\omega|=1}\int\limits_0^d |Du(x+r\omega)|\rho^{n-1}d\rho d\omega dr \\
&= \frac{d^n}{n|S|}\int\limits_0^\infty\int\limits_{|\omega|=1} |Du(x+r\omega)|\,d\omega dr \\
&= \frac{d^n}{n|S|}\int\limits_\Omega |x-y|^{1-n}|Du(y)|dy\,.
\end{aligned}$$

□

Corollary 5.2.3 (Morrey embedding theorem)**.** *Let $u \in W_0^{1,p}(\Omega)$ and $p > n$. Then $u \in C^\gamma(\bar{\Omega})$, where $\gamma = 1-\frac{n}{p}$. Furthermore for any ball $B_R \subset \mathbb{R}^n$, we have*

$$\operatorname{osc}_{\Omega\cap B_R} u \le c(n,p)R^\gamma\|Du\|_{L^p(B_R)}$$

Proof. Since $u \in W_0^{1,p}(\Omega)$ we can extend u by zero on all of $\mathbb{R}^n$ and consider u as an element of $W_0^{1,p}(\mathbb{R}^n)$. By Lemma 5.2.1 with $\mu = \frac{1}{n}$, V_μ maps L^p into L^∞ (since $p > n$). With $\Omega = B_R$ we have

$$\|V_\mu(|Du|)\|_\infty \le \left(\frac{1-\frac{1}{p}}{\frac{1}{n}-\frac{1}{p}}\right)^{1-\frac{1}{p}} \omega_n^{1-\frac{1}{n}}|B_R|^{\frac{1}{n}-\frac{1}{p}}\|Du\|_{L^p(B_R)} \le c(n,p)R^{1-\frac{n}{p}}\|Du\|_{L^p(B_R)}$$

But we have

$$V_\mu(|Du|)(x) = \int\limits_{B_R}|x-y|^{n(1/n-1)}|Du(y)|\,dy = \int\limits_{B_R}|x-y|^{1-n}|Du|(y)\,dy$$

Thus we can apply Lemma 5.2.2 and obtain for a.e. $y \in B_R$:

$$|u(x)-u_{B_R}| \le c(n,p)R^{1-\frac{n}{p}}\|Du\|_{L^p(B_R)}$$

And thus we have for a.e. $y \in B_R$:

$$|u(x)-u(y)| \le |u(x)-u_{B_R}| + |u(y)-u_{B_R}| \le 2c(n,p)R^{1-\frac{n}{p}}\|Du\|_{L^p(B_R)}$$

□

We define the Morrey norm by

$$\|f\|_{M^p(\Omega)} := \inf\left\{K \in \mathbb{R} \,\middle|\, \int\limits_{\Omega\cap B_R(y)} |f(x)|\,dx \le K R^{n(1-\frac{1}{p})} \quad \forall\, B_R(y)\right\}$$

The set

$$M^p(\Omega) := \{f \in L^1(\Omega) \,|\, \|f\|_{M^p(\Omega)} < \infty\}$$

is called the *Morrey-space* and $\|\cdot\|_{M^p(\Omega)}$ is a norm on it. If Ω is bouded, we have by Hölder's inequality that $M^{p'}(\Omega) \subset M^p(\Omega)$ for $p' \ge p$. With the help of the Lebesgue point lemma we have $M^\infty(\Omega) = L^\infty(\Omega)$.

Lemma 5.2.4. *Let $f \in M^p(\Omega), \delta = \frac{1}{p} < \mu$. Then for all $x \in \Omega$ we have*

$$|V_\mu f(x)| \le \frac{1-\delta}{\mu-\delta} d^{n(\mu-\delta)} \|f\|_{M^p(\Omega)}\,.$$

Proof. Fix $x \in \Omega$. We extend f by 0 outside of Ω and write $\nu(\rho) := \int_{B_\rho(x)} |f(y)|dy$. Note the the derivative of ν is just the integral along the boundary. With $K = \|f\|_{M^p}$ we have

$$\begin{aligned}
|V_\mu f(x)| &\le \int\limits_\Omega |x-y|^{n(\mu-1)}|f(y)|dy = \int\limits_0^d \rho^{n(\mu-1)} \int\limits_{\partial B_\rho(x)} |f(y)|dy d\rho \\
&= \int\limits_0^d \rho^{n(\mu-1)}\nu'(\rho)d\rho = d^{n(\mu-1)}\nu(d) + n(1-\mu)\int\limits_0^d \rho^{n(\mu-1)-1}\nu(\rho)d\rho \\
&\le d^{n(\mu-1)}Kd^{n(1-\frac{1}{p})} + n(1-\mu)\int\limits_0^d \rho^{n(\mu-1)-1}K\rho^{n(1-\frac{1}{p})}d\rho \\
&\le Kd^{n(\mu-\frac{1}{p})} + n(1-\mu)\int\limits_0^d \rho^{n(\mu-1)-1}K\rho^{n(1-\frac{1}{p})}d\rho \\
&= Kd^{n(\mu-\frac{1}{p})} + \frac{n(1-\mu)}{n(\mu-\frac{1}{p})}Kd^{n(\mu-\frac{1}{p})} = K\left(\frac{1-\delta}{\mu-\delta}\right)d^{n(\mu-\delta)}\,.
\end{aligned}$$

□

Lemma 5.2.5. *Let $f \in M^p(\Omega)$, $p > 1$, $g := V_\mu f$, $\mu = \frac{1}{p}$. Then there exist constants $c_1, c_2 > 0$, depending only on n and p, such that*

$$\int\limits_\Omega \exp\left(\frac{|g|}{c_1 K}\right)dx \le c_2 d^n$$

with $K = \|f\|_{M^p}$.

Proof. Write for $q \geq 1$ fixed

$$\mu - 1 = \left(1 - \frac{1}{q}\right)(\mu - 1) + \frac{1}{q}(\mu - 1) = \left(1 - \frac{1}{q}\right)(\mu - 1) + \frac{1}{q}\left(\frac{1}{q}\mu + \left(1 - \frac{1}{q}\right)\mu - 1\right)$$
$$= \left(1 - \frac{1}{q}\right)\left(\mu - 1 + \frac{\mu}{q}\right) + \frac{1}{q}\left(\frac{\mu}{q} - 1\right).$$

And thus

$$|x-y|^{n(\mu-1)} = |x-y|^{\frac{n}{q}(\frac{\mu}{q}-1)}|x-y|^{n(1-\frac{1}{q})(\mu-1+\frac{\mu}{q})}$$

so Hölder gives

$$|g(x)| = \left|\int |x-y|^{n(\mu-1)} f(y)dy\right| \leq (V_{\frac{\mu}{q}}|f|)^{\frac{1}{q}} (V_{\mu+\frac{\mu}{q}}|f|)^{1-\frac{1}{q}}$$

By Lemma 5.2.4 (since $\mu = \frac{1}{p} < \mu + \frac{\mu}{q}$) we have

$$V_{\mu+\frac{\mu}{q}}|f| \leq \frac{(1-\mu)q}{\mu} d^{\frac{n}{pq}} K = (p-1)qd^{\frac{n}{pq}}K\,,$$

whereas with Lemma 5.2.1 ($L^1 \to L^1$) we have

$$\int_\Omega V_{\frac{\mu}{q}}|f|dx \leq pq\omega_n^{1-\frac{1}{pq}}|\Omega|^{\frac{1}{pq}}\|f\|_{L^1} \leq pq\omega_n K d^{n(1-\frac{1}{p}+\frac{1}{pq})}\,.$$

and thus

$$\int_\Omega |g|^q \leq p(p-1)^{q-1}\omega_n q^q K^q d^n = \frac{p}{p-1}\omega_n\left((p-1)qK\right)^q d^n$$

So if we choose c_1 such that $(p-1)e < c_1$ we get

$$\int_\Omega \sum_{m=0}^{N} \frac{|g|^m}{m!(c_1K)^m} \leq \frac{p}{p-1}\omega_n d^n \sum_{m=0}^{N}\left(\frac{p-1}{c_1}\right)^m \frac{m^m}{m!} \leq c_2 d^n$$

since (by the Sterling formula) it holds $\frac{m^m}{m!e^m} \approx \frac{1}{\sqrt{2\pi m}} \leq \tilde{c}$ and by the choice of c_1

$$\gamma c_1 = (p-1)e \text{ with } \gamma \in (0,1)$$

it follows that

$$\left(\frac{p-1}{c_1}\right)^m \frac{m^m}{m!} \leq \left(\frac{\gamma}{e}\right)^m \frac{m^m}{m!} \leq \tilde{c}\gamma^m\,.$$

□

Theorem 5.2.6 (John-Nirenberg estimate)**.** *Let $u \in W^{1,1}(\Omega)$, Ω convex, and suppose there is a $K > 0$ such that*

$$\int_{\Omega \cap B_R} |Du|\, dx \leq K R^{n-1} \quad \forall\, B_R \tag{5.12}$$

that is $|Du| \in M^n(\Omega)$. Then there are constants σ_0, C, depending only on n such that

$$\int_\Omega \exp\left(\frac{\sigma}{K}|u - u_\Omega|\right) dx \leq C d^n$$

with $\sigma = \sigma_0 |\Omega| d^{-n}$.

Proof. By Lemma 5.2.2 we have

$$|u(x) - u_\Omega| \leq \frac{d^n}{n|\Omega|} \int |x-y|^{1-n} |Du(y)| dy = \frac{d^n}{n|\Omega|} V_{\frac{1}{n}}(|Du|)(x)$$

By (5.12) we have $|Du| \in M^n(\Omega)$ and thus Lemma 5.2.5 gives the result. □

5.3 Hölder continuity

With Theorem 5.1.3, the weak Harnack inequality, we can prove a strong maximum principle for weak solutions:

Theorem 5.3.1. *Let the operator L be strictly elliptic, i.e. $a^{ij} \geq \lambda \cdot \delta^{ij}$ for a $\lambda > 0$ and $\sum_{i,j}(a^{ij})^2 \leq \Lambda^2$,*

$$\lambda^{-2}\left(\sum_i \left|b^i\right|^2 + \left|c^i\right|^2\right) + \lambda^{-1}|d| \leq \nu^2$$

with

$$\int_\Omega (dv - b^i D_i v)\, dx \leq 0\,, \tag{5.13}$$

for every $v \in C_0^1(\Omega)$, $v \geq 0$. Assume furthermore that $u \in W^{1,2}(\Omega)$ satisfies $Lu \geq 0$. If

$$\sup_B u = \sup_\Omega u \geq 0$$

for some ball $B \Subset \Omega$, then u must be constant and we have equality in (5.13) if $u \neq 0$.

Proof. Since u is a subsolution and $u \in W^{1,2}(\Omega)$ we have by Remark 5.1.5, that $\sup_B u$ is finite. By shrinking the ball $B = B_R(y)$, if necessary, we can assume that $B_{4R}(y) \subset \Omega$. We define

$$M = \sup_{\Omega} u = \sup_{B_R(y)} u \geq 0$$

and note that $v := M - u$ is a positive supersolution. Recall that

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^i D_i u + du.$$

and thus

$$LM = D_i(b^i M) + dM\,.$$

Since is M is non-negative we have by (5.13) that $LM \leq 0$ and thus $L(M - u) \leq 0$.

Applying Theorem 5.1.3 with $p = 1$, we have

$$R^{-n} \int\limits_{B_{2R}(y)} M - u\, dx \leq C \inf_{B_R(y)} (M - u) = 0$$

and thus $u \equiv M$ on $B_{2R}(y)$. Therefore the set $\{u = M\}$ is open. That this set is also relatively closed follows analogously: Choose a point in $p \in \partial\{u = M\} \cap \Omega$. We can assume that $B_\delta(p) \Subset \Omega$. Since $\{u = M\}$ is open, we have $\sup_{B_\delta(p)} u = M$ and thus by the same argument that $u \equiv M$ on $B_{2\delta}(p)$, which implies that $u(p) = M$. Thus $u = M$ on Ω.

Since u is a subsolution and $u \equiv M$, we get $LM \geq 0$ which implies

$$M \int\limits_{\Omega} (b^i D_i v - dv)\, dx \leq 0$$

for all $v \in C_0^1(\Omega)$, $v \geq 0$ and thus in case $M \neq 0$ that equality holds in (5.13). ☐

Theorem 5.3.2 (Harnack inequality)**.** *Assume L is as in Theorem 5.3.1, and $u \in W^{1,2}(\Omega)$, $u \geq 0$, satisfies $Lu = 0$. Then for any ball $B_{4R}(y) \subset \Omega$, we have*

$$\sup_{B_R} u \leq C \inf_{B_R} u$$

where $C = C(n, \Lambda/\lambda, \nu, R)$.

Proof. This follows from Theorems 5.1.2 and 5.1.3. Note that $K(R) = 0$ and $u \geq 0$ on $B_{4R}(y)$, so Theorem 5.1.2 gives

$$\sup_{B_R(y)} u \leq C \cdot R^{-n/2} \|u\|_{L^2(B_{2R}(y))}$$

and Theorem 5.1.3 yields

$$C \cdot R^{-n/2} \|u\|_{L^2(B_{2R}(y))} \leq C \cdot \inf_{B_R(y)} u\,.$$

□

Corollary 5.3.3. *Let L and u be as in Theorem 5.3.2, then for every domain $\Omega' \Subset \Omega$ we have*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u\,,$$

where $C = C(\Omega, \Omega', n, \nu, \Lambda/\lambda)$.

Proof. Pick R such that $0 < R < \frac{1}{4}\operatorname{dist}(\Omega', \partial\Omega)$. Since Ω' precompact we can cover Ω' by a finite number N of balls $B_R(y_i)$, i.e.

$$\Omega' \subseteq \bigcup_{i=1}^{N} B_R(y_i)\,,$$

where by the choice of R we have as well $B_{4R}(y_i) \subset \Omega$. Let x_1, $x_2 \in \overline{\Omega'}$ s.t. $u(x_1) = \inf_{\Omega'} u$, $u(x_2) = \sup_{\Omega'} u$, and choose a closed arc $\Gamma \subseteq \bar{\Omega}'$, joining x_1 and x_2. Again Γ is covered by a subcollection of the balls $B_R(y_i)$. We apply Theorem 5.3.2 on every such ball (note that N is independent of Γ) to arrive at

$$u(x_2) \leq C^N u(x_1)\,.$$

□

The estimate from Theorem 5.1.3 allows us to show that solutions of $Lu = D_i f^i + g$ are Hölder continous.

Theorem 5.3.4. *Let the operator L be strictly elliptic, i.e. $a^{ij} \geq \lambda \cdot \delta^{ij}$ for a $\lambda > 0$ and $\sum_{i,j}(a^{ij})^2 \leq \Lambda^2$,*

$$\lambda^{-2}\left(\sum_i \left|b^i\right|^2 + \left|c^i\right|^2\right) + \lambda^{-1}\left|d\right| \leq \nu^2\,.$$

Assume further that $f^i \in L^q(\Omega)$ for $1 \leq i \leq n$ and $g \in L^{q/2}(\Omega)$ for a $q > n$. If $u \in W^{1,2}(\Omega)$ solves $Lu = D_i f^i + g$, then u is locally Hölder continuous in Ω, and for every ball $B_0 = B_{R_0}(y) \subset \Omega$ and $R \leq R_0$ we have

$$\operatorname{osc}_{B_R(y)} u \leq C \cdot R^{\alpha} \cdot \left(R_0^{-\alpha} \sup_{B_0} |u| + k\right)$$

where $C = C(n, \Lambda/\lambda, \nu, q, R_0)$ and $\alpha = \alpha(n, \Lambda/\lambda, \nu R_0, q)$ are positive constants and $k = \frac{1}{\lambda}(\|f\|_q + \|g\|_{\frac{q}{2}})$.

Proof. Assume w.l.o.g. that $R \le R_0/4$. We can also assume that $\sup_{B_0} |u| < \infty$, otherwise there is nothing to prove. We write

$$\begin{aligned} M_0 &= \sup_{B_0} |u| \\ M_4 &= \sup_{B_{4R}} u \\ M_1 &= \sup_{B_R} u \\ m_1 &= \inf_{B_R} u \\ m_4 &= \inf_{B_{4R}} u, \end{aligned}$$

then $M_4 - u \ge 0$ on B_{4R}, as well as $u - m_4 \ge 0$. Furthermore,

$$\begin{aligned} L(M_4 - u) &= M_4(D_i b^i + d) - D_i f^i - g \\ L(u - m_4) &= -m_4(D_i b^i + d) + D_i f^i + g\,. \end{aligned}$$

Let $\delta = 1 - \frac{n}{q}$ and

$$\bar{k}(R) = \frac{1}{\lambda} \cdot R^{\delta} \cdot \left(\|f\|_{q,B_0} + M_0 \|b\|_{q,B_0} \right) + \frac{1}{\lambda} \cdot R^{2\delta} \cdot \left(\|g\|_{q/2,B_0} + M_0 \|d\|_{q/2,B_0} \right).$$

By the weak Harnack inequality, Theorem 5.1.3, applied on B_{4R} with $p = 1$ we obtain

$$\begin{aligned} R^{-n} \int_{B_{2R}} (M_4 - u)\, dx &\le C\big(M_4 - M_1 + \bar{k}(R)\big) \\ R^{-n} \int_{B_{2R}} (u - m_4)\, dx &\le C\big(m_1 - m_4 + \bar{k}(R)\big)\,. \end{aligned}$$

Adding the inequalities gives

$$M_4 - m_4 \le C'\big(M_4 - m_4 + m_1 - M_1 + \bar{k}(R)\big)$$

thus

$$\operatorname{osc}_{B_R} u = M_1 - m_1 \le \left(1 - \frac{1}{C'}\right)(M_4 - m_4) + \bar{k}(R) = \left(1 - \frac{1}{C'}\right) \operatorname{osc}_{B_{4R}} u + \bar{k}(R)\,.$$

Writing $\omega(R) = \operatorname{osc}_{B_R} u$, we get

$$\omega(R) \le \gamma \omega(4R) + \bar{k}(R)\,,$$

where $C' = C'(n, \Lambda/\lambda, \nu R_0, q)$ and $\gamma = 1 - \frac{1}{C'}$. Note that $0 < \gamma < 1$. To complete the proof, we need the following lemma:

Lemma 5.3.5. *Let $\omega : (0, R_0] \to \mathbb{R}$ be a non-decreasing function, as well as $\sigma : (0, R_0] \to \mathbb{R}$ be non-decreasing such that for some $0 < \tau, \gamma < 1$ it holds*

$$\omega(\tau R) \leq \gamma\omega(R) + \sigma(R)$$

for all $R \leq R_0$. Then for any $R \leq R_0$ and $0 < \mu < 1$ it holds that

$$\omega(R) \leq C \cdot \left(\left(\frac{R}{R_0} \right)^{\alpha} \omega(R_0) + \sigma(R^{\mu} R_0^{1-\mu}) \right)$$

where $C = C(\gamma, \tau)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive contants.

Proof. Fix some $0 < R_1 < R_0$. By the monotonicity of σ we have for $R \leq R_1$, that

$$\omega(\tau R) \leq \gamma\omega(R) + \sigma(R_1)$$

By iterating this we get for $m \in \mathbb{N}$, that

$$\begin{aligned}
\omega(\tau^m R_1) &\leq \gamma\omega(\tau^{m-1} R_1) + \sigma(R_1) \\
&\leq \gamma^2\omega(\tau^{m-2} R_1) + (1+\gamma)\sigma(R_1) \\
&\leq \quad \cdots \\
&\leq \gamma^m\omega(R_1) + \sum_{i=0}^{m-1} \gamma^i \cdot \sigma(R_1) \\
&\leq \gamma^m\omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}
\end{aligned}$$

For any $R \leq R_1$ there is a $m \in \mathbb{N}$ such that

$$\tau^m R_1 \leq R < \tau^{m-1} R_1 \,,$$

which implies

$$\begin{aligned}
\omega(R) &\leq \omega(\tau^{m-1} R_1) \\
&\leq \gamma^{m-1}\omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}
\end{aligned}$$

and since $\tau^m \leq R/R_1$

$$\begin{aligned}
\log \tau^m &\leq \log R - \log R_1 \\
\iff m &\geq \frac{\log R - \log R_1}{\log \tau}
\end{aligned}$$

and thus

$$\begin{aligned}\gamma^m &= e^{m\log\gamma}\\ &\le e^{\frac{\log R-\log R_1}{\log\tau}\cdot\log\gamma}\\ &= \left(\frac{R}{R_1}\right)^{\frac{\log\gamma}{\log\tau}}\end{aligned}$$

which yields

$$\omega(R)\le\frac{1}{\gamma}\cdot\left(\frac{R}{R_1}\right)^{\frac{\log\gamma}{\log\tau}}\omega(R_0)+\frac{\sigma(R_1)}{1-\gamma}.$$

Now let $R_1=R_0^{1-\mu}R^\mu$, so we get

$$\omega(R)\le\frac{1}{\gamma}\cdot\left(\frac{R}{R_0}\right)^{(1-\mu)\frac{\log\gamma}{\log\tau}}\omega(R_0)+\frac{\sigma(R_0^{1-\mu}R^\mu)}{1-\gamma}.\qquad\square$$

Back to the proof of the theorem. We note that we have in the case at hand

$$\begin{aligned}\sigma(R_0^{1-\mu}R^\mu) &= \bar{k}(R_0^{1-\mu}R^\mu)\\ &= \frac{1}{\lambda}\cdot R^{\delta\mu}R_0^{\delta(1-\mu)}\cdot\big(\|f\|_{q,B_0}+M_0\,\|b\|_{q,B_0}\big)\\ &\qquad+\frac{1}{\lambda}R^{2\delta\mu}R_0^{2\delta(1-\mu)}\cdot\big(\|g\|_{q/2,B_0}+M_0\,\|d\|_{q/2,B_0}\big)\\ &\le\left(\frac{R}{R_0}\right)^{\mu\delta}\cdot\Bigg(\frac{1}{\lambda}\cdot R_0^\delta\cdot\big(\|f\|_{q,B_0}+M_0\,\|b\|_{q,B_0}\big)\\ &\qquad+\frac{1}{\lambda}R_0^{2\delta}\cdot\big(\|g\|_{q/2,B_0}+M_0\,\|d\|_{q/2,B_0}\big)\Bigg)\end{aligned}$$

Now choose μ such that

$$\alpha:=(1-\mu)\frac{\log\gamma}{\log\tau}<\mu\delta\,,$$

then

$$(R/R_0)^{\mu\delta}\le(R/R_0)^\alpha$$

and thus

$$\operatorname{osc}_{B_R}u\le C\cdot\left(\frac{R}{R_0}\right)^\alpha\cdot\big(\sup_{B_0}|u|+k\big).\quad\square$$

Using Remark 5.1.5 we can bound $\sup |u|$ by the L^2-norm of u.

Theorem 5.3.6. *Let L be as in the previous theorem, $f^i \in L^q(\Omega)$, $g \in L^{\frac{q}{2}}(\Omega)$, $q > n$. If $Lu = D_i f^i + g$ in Ω, then we have for every $\Omega' \subset\subset \Omega$, that*

$$\|u\|_{C^\alpha(\Omega')} \le C(\|u\|_{L^2(\Omega)} + k)$$

with $C = C(n, \frac{\Lambda}{\lambda}, \nu, q, d')$, $d' = \operatorname{dist}(\Omega', \partial\Omega)$, $\alpha = \alpha(n, \frac{\Lambda}{\lambda}, \nu, d') > 0$ and $k = \frac{1}{\lambda}(\|f\|_{L^q(\Omega)} + \|g\|_{L^{\frac{q}{2}}(\Omega)})$.

Proof. We use $R_0 = d'/4$ in Theorem 5.3.4 and Remark 5.1.5 to estimate $\sup |u|$. □

5.4 Estimates at the boundary

We will now briefly sketch how to derive Hölder estimates up to the boundary. Let T be any subset of $\overline{\Omega}$. We will say that $u \le 0$ on T for a $W^{1,2}$-function u, if u^+ is the limit of a sequence of non-negative functions in $C^1_c(\overline{\Omega} \setminus T)$.

Theorem 5.4.1. *We assume that L satisfies the same structural conditions as before, $f^i \in L^q(\Omega)$, $g \in L^{\frac{q}{2}}(\Omega)$, for some $q > n$. Then if $u \in W^{1,2}(\Omega)$ is a subsolution of $Lv = D_i f^i + g$ in Ω, we have for any $y \in \mathbb{R}^n$, $R > 0$ und $p > 1$:*

$$\sup_{B_R(y)} u^+_M \le C(R^{-\frac{n}{p}} \|u^+_M\|_{L^p(B_{2R}(y))} + k(R))$$

where

$$M = \sup_{\partial\Omega \cap B_{2R}(y)} u^+$$

$$u^+_M = \begin{cases} \sup\{u(x), M\} & , x \in \Omega \\ M & , x \notin \Omega \end{cases}$$

and $k(R) = \lambda^{-1}(R^\delta \|f\|_{L^q(B_{4R}(y)} + R^{2\delta}\|g\|_{L^{\frac{q}{2}}(B_{4R}(y))})$, $\delta = 1 - \frac{n}{q}$, $C = C(n, \frac{\Lambda}{\lambda}, \nu R, q, p)$.

Theorem 5.4.2. *Let the conditions for L, f^i and g be as above, $u \in W^{1,2}(\Omega)$, $Lu \le D_i f^i + g$, $u \ge 0$ on $B_{4R}(y) \cap \overline{\Omega} \subset \mathbb{R}^n$. Then for every p, $1 \le p < \frac{n}{n-2}$ we have*

$$R^{-\frac{n}{p}} \|u^-_m\|_{L^p(B_{2R}(y))} \le C(\inf_{B_R(y)} u^-_m + k(R))$$

where

$$m = \inf_{\partial\Omega \cap B_{4R}(y)} u$$

$$u^-_m = \begin{cases} \inf\{u(x), m\} & , x \in \Omega \\ m & , x \notin \Omega \end{cases}$$

and $C = C(n, \frac{\Lambda}{\lambda}, \nu R, q, p)$.

Proof. Let

$$\overline{u} = u_M^+ + K \text{ for Theorem 5.4.1}\,,$$
$$\overline{u} = u_m^- + K \text{ for Theorem 5.4.2}\,.$$

and define the testfunctions

$$v = \eta^2 \begin{cases} \overline{u}^\beta - (M+K)^\beta & \text{if } \beta > 0 \\ \overline{u}^\beta - (m+K)^\beta & \text{if } \beta < 0 \end{cases}$$

with $\eta \in C_0^1(B_{4R}(y))$. Then one works analogously as in the proofs of Theorems 5.1.5 und 5.1.3. □

Theorem 5.4.3. *Let L be as above, $f^i \in L^q(\Omega)$, $g \in L^{\frac{q}{2}}(\Omega)$ for some $q > n$. If $u \in W^{1,2}(\Omega)$ satisfies $Lu = D_i f^i + g$ in Ω and Ω satisfies an exteriour cone condition at a point $x_0 \in \partial\Omega$ then we have for any $0 < R < R_0$ and $B_0 = B_{R_0}(x_0)$:*

$$\operatorname{osc}_{B_R \cap \Omega} u \le C(R^\alpha (R_0^{-\alpha} \sup_{\Omega \cap B_0} |u| + k) + \sigma(\sqrt{RR_0}))$$

where $\sigma(R) = \operatorname{osc}_{\partial\Omega \cap B_R(x_0)} u$ and $C = C(n, \frac{\Lambda}{\lambda}, \nu, q, R_0, V_{x_0})$, $\alpha = \alpha(n, \frac{\Lambda}{\lambda}, \nu R_0, q, V_{x_0})$ are positive constants and V_{x_0} is the exteriour cone to Ω at x_0.

Proof. We follow the proof of Theorem 5.3.4. We can again asumme w.l.o.g. that $R \le min\{\frac{R_0}{4}, \text{height} V_{x_0}\}$ and write

$$M_0 = \sup_{\Omega \cap B_{R_0}} |u| \qquad M_1 = \sup_{\Omega \cap B_R} u$$
$$m_1 = \inf_{\Omega \cap B_R} u \qquad M_4 = \sup_{\Omega \cap B_{4R}} u$$
$$m_4 = \inf_{\Omega \cap B_{4R}} u$$
$$M = \sup_{\partial\Omega \cap B_{4R}} u \qquad m = \inf_{\partial\Omega \cap B_{4R}} u\,.$$

Then by Theorem 5.4.2, applied to $M_4 - u$ and $u - m_4$ we obtain

$$\begin{aligned}(M_4 - M)\frac{|B_{2R}(x_0)\setminus\Omega|}{R^n} &\le R^{-n}\int_{B_{2R}(x_0)} (M_4-u)^-_{M_4-M}\,dx \\ &\le C(\inf_{B_R(x_0)}(M_4-u)^-_{M_4-M} + \bar{k}(R)) \le C(M_4 - M_1 + \overline{k}(R)) \\ (m - m_4)\frac{|B_{2R}(x_0)\setminus\Omega|}{R^n} &\le R^{-n}\int_{B_{2R}(x_0)} (u-m_4)^-_{m-m_4}dx \\ &\le C(\inf_{B_R(x_0)}(u-m_4)^-_{m-m_4} + \bar{k}(R)) \le C(m_1 - m_4 + \overline{k}(R)),\end{aligned}$$

where

$$\bar{k}(R) = \frac{1}{\lambda}\cdot R^{\delta}\cdot\big(\|f\|_{q,B_{R_0}} + M_0\,\|b\|_{q,B_{R_0}}\big) + \frac{1}{\lambda}\cdot R^{2\delta}\cdot\big(\|g\|_{q/2,B_{R_0}} + M_0\,\|d\|_{q/2,B_{R_0}}\big).$$

Using the exteriour cone condition, which implies that $|B_{2R}(x_0)\setminus\Omega| \ge \delta(R)^n$ for a $\delta > 0$ and obtain

$$\begin{aligned}M_4 - M &\le C(M_4 - M_1 + \overline{k}(R)),\\ m - m_4 &\le C(m_1 - m_4 + \overline{k}(R)).\end{aligned}$$

Adding these two inequalities gives:

$$\operatorname{osc}_{\Omega\cap B_R(x_0)} u \le \gamma\ \operatorname{osc}_{\Omega\cap B_{4R}(x_0)} u + 2\overline{k}(R) + \operatorname{osc}_{\partial\Omega\cap B_{4R}} u$$

where $\gamma = 1 - \frac{1}{C}$, $C = C(n, \frac{\Lambda}{\lambda}, \nu R_0, q, V_{x_0})$. Then apply again Lemma 5.3.3. □

Theorem 5.4.4. *Let the operator L be as before, $f^i \in L^q(\Omega)$, $g \in L^{\frac{q}{2}}(\Omega)$, for some $q > n$ and assume that Ω satisfies a uniform exterior cone condition. Then if $u \in W^{1,2}(\Omega)$ solves $Lu = D_i f^i + g$ in Ω with constants $K, \alpha_0 > 0$ s.t.*

$$osc_{\partial\Omega\cap B_R(x_0)} u \le K R^{\alpha_0} \quad \forall x_0 \in \partial\Omega,\ R > 0,$$

then $u \in C^{\alpha}(\overline{\Omega})$ for some $\alpha > 0$ and

$$\|u\|_{C^{\alpha}(\overline{\Omega})} \le C(\sup_{\Omega}|u| + K + k)$$

where $\alpha = \alpha(n, \frac{\Lambda}{\lambda}, \nu \operatorname{diam}\Omega, q, V, \alpha_0)$, $C = C(n, \frac{\Lambda}{\lambda}, \nu, q, V, \alpha_0)$ and $k = \lambda^{-1}(\|f\|_{q,\Omega} + \|g\|_{q/2,\Omega}$.

Proof. Let $y \in \Omega$, $\delta = \operatorname{dist}(y, \partial\Omega)$. By Theorem 5.3.4 with $R_0 = \delta$ we have for any $x \in B_\delta(y)$:

$$\frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \le c(\delta^{-\alpha}\sup_{B_\delta}|u| + k)$$

Choose $x_0 \in \partial\Omega$ s.t. $\delta = |y - x_0|$. By Theorem 5.4.3 with $R = 2\delta$, $R_0 = \operatorname{diam}\Omega$ we obtain

$$\begin{aligned}
&\delta^{-\alpha} osc_{B_\delta(y)} u \le \delta^{-\alpha} \operatorname{osc}_{\Omega\cap B_{2\delta}(x_0)} u \\
&\le \delta^{-\alpha}(C(2\delta)^\alpha((\operatorname{diam}\Omega)^{-\alpha}\sup_\Omega |u| + k) + K(2\delta)^{\frac{\alpha_0}{2}}(\operatorname{diam}\Omega)^{\frac{\alpha_0}{2}}) \\
&\le C(\sup_\Omega |u| + K + k)
\end{aligned}$$

if $2\alpha \le \alpha_0$. By working with $v := u - u(y)$, i.e. replacing $k(R)$ by

$$\overline{k}(R) = \frac{1}{\lambda}(\|f\|_{L^q} + C\sup_\Omega |u| + \|g\|_{L^{\frac{q}{2}}})$$

where $C = C(\nu, \operatorname{diam}\Omega, q, n)$ we can assume that $u(y) = 0$. This implies

$$\delta^{-\alpha} \sup_{B_\delta(y)} |u| \le \delta^{-\alpha} \operatorname{osc}_{B_\delta(y)} u$$

and thus

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \le C(\sup_\Omega |u| + k + K)\,.$$

Now assume that $|x - y| \ge \delta$. Then by Theorem 5.4.3 with the same x_0 as before, $R = 2|x - y|$, $R_0 = \operatorname{diam}\Omega$ we have for $\alpha \le 2\alpha_0$:

$$\begin{aligned}
\frac{|u(x) - u(y)|}{|x - y|^\alpha} &\le C((\operatorname{diam}\Omega)^{-\alpha}\sup_\Omega |u| + k + K(\operatorname{diam}\Omega)^{\frac{\alpha_0}{2}}) \\
&\le C(\sup_\Omega |u| + k + K)
\end{aligned}$$

□

6 Sup- and gradient bounds for quasilinear elliptic equations

We will treat the following initial value problem for a graphical surface of prescribed mean curvature: Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, $\partial\Omega$ of class $C^{2,\alpha}$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$.

$$\begin{cases} \mathcal{A}u = \operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = \mathcal{H}(x, u(x)) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

where $\mathcal{H} : \Omega \times \mathbb{R} \to \mathbb{R}$ is of class C^2 and $\frac{\partial \mathcal{H}}{\partial z} \geq 0$.

Remark 6.0.1: We have already seen before that solutions to (6.1) are unique. To apply the continuity method to show existence we need to establish a-priori estimates in $C^{1,\alpha}(\overline{\Omega})$ for solutions of (6.1).

6.1 The sup-estimate

We will need a further assumption to establish a sup bound. Let $\mathcal{A}u = \mathcal{H}(x)$ in Ω, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, where we assume for the moment that $\mathcal{H}(\cdot)$ does not depend on u. We compute

$$\begin{aligned} \int_\Omega \mathcal{H}(x)\, dx &= \int_\Omega D_i(A^i(Du))\, dx = \int_{\partial\Omega} A^i(Du) \cdot \nu_i \, d\sigma \\ &= \int_{\partial\Omega} \frac{D_i u}{\sqrt{1+|Du|^2}} \cdot \nu_i \, d\sigma \quad \begin{cases} < |\partial\Omega| \, , \\ > -|\partial\Omega| \, , \end{cases} \end{aligned}$$

since

$$\left| \frac{D_i u}{\sqrt{1+|Du|^2}} \right| < 1 \, .$$

This implies

$$\left| \int_\Omega \mathcal{H}(x)\, dx \right| < |\partial\Omega| \, .$$

This is a necessary condition on the data! Even more, multiplying with a test function $\eta \in C^1_c(\Omega)$ and integrating we get the necessary condition:

$$\left|\int_\Omega \mathcal{H}\eta\,dx\right| = \left|\int_\Omega A^i(Du)D_i\eta\,dx\right| < \int_\Omega |D\eta|\;dx\,.$$

Slightly strenghtening this condition leads to a sufficient condition to ensure a sup-bound. We again allow that the right hand side H depends on x and u and assume further that there exists an $\varepsilon_0 > 0$ such that

$$\left|\int_\Omega \mathcal{H}(x,0)\eta\,dx\right| \le (1-\varepsilon_0)\int_\Omega |D\eta|dx \quad \forall \eta \in C^1_c(\Omega). \tag{6.2}$$

For the proof we need the following technical lemma:

Lemma 6.1.1 (Stampacchia)**.** *If $\varphi : [0,\infty) \to \mathbb{R}^+$ is monotonically decreasing and there are constants $\gamma > 1$ and $0 < C_0 < \infty,\ p \ge 1$ with*

$$|h-k|^p\varphi(h) \le C_0\,(\varphi(k))^\gamma \quad \forall\, h > k \ge k_0, \tag{6.3}$$

then $\varphi(k_0+d) = 0$, where $d^p = C_0 \cdot 2^{\frac{p\gamma}{\gamma-1}}\varphi(k_0)^{\gamma-1}$.

Proof. Set $k_r = k_0 + d - \left(\frac{d}{2^r}\right)$ for $r = 0,1,2,\ldots$. Then k_r is incresing in r. From (6.3) we have

$$\varphi(k_{r+1}) \le C_0\left(\frac{2^{(r+1)p}}{d^p}\right)\varphi(k_r)^\gamma.$$

We now claim:

$$\varphi(k_r) \le \varphi(k_0)\cdot 2^{r\mu} \quad \text{where } \mu = \frac{p}{1-\gamma} < 0. \tag{6.4}$$

To prove this claim we apply induction. The case $r=0$ is trivial. We assume that (6.4) is true for r and consider the case $r+1$:

$$\begin{aligned}
\varphi(k_{r+1}) &\overset{(6.3)}{\le} C_0\left(\tfrac{2^{(r+1)p}}{d^p}\right)\varphi(k_r)^\gamma\\
&\overset{(6.4)}{\le} C_0\left(\tfrac{2^{(r+1)p}}{d^p}\right)\varphi(k_0)^\gamma\cdot 2^{r\mu\gamma}\\
&= C_0\left(\tfrac{2^{(r+1)p}}{d^p}\right)\varphi(k_0)^\gamma\cdot 2^{\frac{rp\gamma}{1-\gamma}}\\
&= 2^{(r+1)p}\cdot 2^{\frac{rp\gamma}{1-\gamma}}\cdot 2^{\frac{-p\gamma}{\gamma-1}}\varphi(k_0).
\end{aligned}$$

We compute the exponent

$$(r+1)p - rp\frac{\gamma}{\gamma-1} - p\frac{\gamma}{\gamma-1} = (r+1-(r+1)\frac{\gamma}{\gamma-1})p = (r+1)\frac{1}{1-\gamma}p = \mu(r+1)\,.$$

Thus (6.4) is true. We now take $r \to \infty$ in (6.4). Since $\mu < 0$, we obtain

$$\varphi(k_0 + d) = 0\,.$$

□

Proposition 6.1.2. *If the condition* (6.2) *holds, then any solution* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *of* (6.1) *satisfies an estimate of the form*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\varphi| + C(n, \varepsilon_0^{-1}, |\Omega|)$$

Proof. We want to use a Stampacchia iteration. Define

$$u_k = \max(u - k, 0) \ \text{ for } k > k_0 = \max\{\sup_{\partial\Omega} \varphi, 0\}\ .$$

Then $u_k|_{\partial\Omega} = 0$ and $u_k \in C^{0,1}(\bar{\Omega}) = W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$. The function u_k lives on the set

$$A(k) = \{x \in \Omega \mid u(x) > k\}.$$

Note that $|A(k)|$ is monotonically decreasing in k. Since $u_k \in W_0^{1,2}$ we can multiply $\mathcal{A}u = \mathcal{H}(x, u)$ with the testfunction u_k and integrate by parts. We obtain

$$-\int_{\Omega} \frac{D_i u}{\sqrt{1 + |Du_k|^2}} \cdot D_i u_k \, dx = \int_{\Omega} \mathcal{H}(x, u) u_k \, dx\,.$$

If $u_k(x) > 0$ we have that $D_i u_k(x) = D_i u(x)$. So we can estimate

$$\begin{aligned} \int_{A(k)} \frac{|Du_k|^2}{\sqrt{1 + |Du_k|^2}} \, dx &= -\int_{A(k)} \mathcal{H}(x, u) u_k \, dx \\ &\leq -\int_{A(k)} \mathcal{H}(x, 0) u_k \, dx, \end{aligned}$$

since H was assumed to be monotonically increasing in u. We apply (6.2) and obtain

$$\int_{\Omega} \frac{|Du_k|^2}{\sqrt{1 + |Du_k|^2}} \, dx \leq (1 - \varepsilon_0) \int_{A(k)} |Du_k| \, dx\,.$$

This implies

$$\begin{aligned}\int\limits_{A(k)} |Du_k|\,dx &\le \int\limits_{A(k)} \sqrt{1+|Du_k|^2}\,dx \\ &= \int\limits_{A(k)} \frac{1+|Du_k|^2}{\sqrt{1+|Du_k|^2}}\,dx \\ &\le \int\limits_{A(k)} \underbrace{\frac{1}{\sqrt{1+|Du_k|^2}}}_{\le 1}\,dx + (1-\varepsilon_0)\int\limits_{A(k)} |Du_k|\,dx\,,\end{aligned}$$

which gives

$$\int\limits_{A(k)} |Du_k|\,dx \le \frac{1}{\varepsilon_0}|A(k)| \quad \forall k \ge k_0\ .$$

The Sobolev inequality thus implies

$$\Bigg(\int\limits_{A(k)} |u_k|^{\frac{n}{n-1}dx}\Bigg)^{\frac{n-1}{n}} \le \frac{C(n)}{\varepsilon_0}|A(k)|.$$

Together with the Hölder inequality we see that

$$\int\limits_{A(k)} |u_k|\,dx \le \Bigg(\int\limits_{A(k)} |u_k|^{\frac{n}{n-1}}\,dx\Bigg)^{\frac{n-1}{n}} |A(k)|^{\frac{1}{n}} \le \frac{C(n)}{\varepsilon_0}|A(k)|^{1+\frac{1}{n}}.$$

For $h > k \ge k_0$ we have

$$\begin{aligned}\int\limits_{A(k)} |u_k|\,dx = \int\limits_{A(k)} u_k\,dx \ge \int\limits_{A(h)} u_k\,dx &= \int\limits_{A(h)} (u-k)\,dx \\ &\ge \int\limits_{A(h)} (h-k)\,dx = |h-k||A(h)|\,.\end{aligned}$$

With the estimate before this implies

$$|h-k||A(h)| \le \varepsilon_0^{-1}C(n)|A(k)|^{1+\frac{1}{n}}.$$

Thus condition (6.3) is fulfilled and Lemma 6.1.1 gives that

$$|A(k_0+d)| = 0, \quad d = \tilde{C}(n)\varepsilon_0^{-1}|A(k_0)|^{\frac{1}{n}} \le \tilde{C}(n)\varepsilon_0^{-1}|\Omega|^{\frac{1}{n}}.$$

This implies

$$\sup_{\Omega} u \le \sup_{\partial\Omega} \varphi + \tilde{C}(n)\varepsilon_0^{-1}|\Omega|^{\frac{1}{n}}$$

To estimate $\inf_\Omega u$ we look at $v := -u$. Note that v solves $\mathcal{A}v = -\mathcal{H}(x,-v) =: \tilde{\mathcal{H}}(x,v)$. Note that $\frac{\partial \tilde{\mathcal{H}}}{\partial z} \ge 0$, so we can proceed as before. □

6.2 A maximum principle for the gradient

Recall: If $\Delta u = 0$, then also $\Delta(D_l u) = D_l \Delta u = 0$. Thus

$$\begin{aligned} 0 &= D^l u \Delta D^l u = D^l u D^i D_i D_l u = D_i(D^l u D^i D_l u) - D_i D^l u D^i D_l u \\ &= D_i(D^i \underbrace{(D^l u D_l u)}_{=|Du|^2}) - D^i(D_i D^l u D_l u) - D_i D^l u D^i D_l u \\ &= \Delta |Du|^2 - \underbrace{\Delta D^l u}_{=0} D_l u - 2 D_i D^l u D^i D_l u = \Delta |Du|^2 - 2|D^2 u|^2 \end{aligned}$$

This implies

$$\Delta |Du|^2 = 2|D^2 u|^2 \geq 0$$

so by the maximum principle

$$\max_{\Omega} |Du|^2 \leq \max_{\partial\Omega} |Du|^2 .$$

In the case of prescribed mean curvature we have the following:

$$D_i(A^i(Du)) = \mathcal{H} .$$

Differentiating in direction D_j gives

$$D_i(a^{ik}(Du) D_j D_k u) = D_j \mathcal{H},$$

where (a^{ik}) is a positive definite, symmetric matrix. We multiply with $D^j u$, and sum over j to get

$$\begin{aligned} & D^j u \cdot D_i(a^{ik}(Du) D_j D_k u) && = D_j \mathcal{H} D^j u \\ \Leftrightarrow \quad & D_i(a^{ik}(Du) \underbrace{D^j u D_j D_k u}_{=\frac{1}{2} D_k |Du|^2}) - a^{ik}(Du) D_i D^j u D_k D_j u && = D_j \mathcal{H} D^j u \end{aligned}$$

This implies

$$\frac{1}{2} D_i(a^{ik}(Du) D_k(|Du|^2)) = a^{ik}(Du) D_i D_j u D_k D_j u + D_j \mathcal{H} D^j u .$$

Let $x \in \Omega$ be fixed. Then $a_{ik}(Du(x))$ is symmetric and positive definite. Thus we can rotate the coordinate system of $\mathbb{R}^n$ such that

$$a^{ik}(Du(x)) = \begin{pmatrix} \lambda^1(x) & & \\ & \ddots & \\ & & \lambda^n(x) \end{pmatrix}$$

This implies $a^{ik}(Du(x)) \geq \lambda_0 \delta^{ik}$ for a $\lambda_0 > 0$ and

$$\begin{aligned} a^{ik}(Du(x))D_jD_iu(x)D_jD_ku(x) &= \sum_{i,j,k} \delta^{ik}\lambda_i(x)D_jD_iu(x)D_jD_ku(x) \\ &= \sum_{i,j} \lambda_i(x)(D_iD_ju(x))^2 \\ &\geq \lambda_0 |D^2u|^2 \geq 0. \end{aligned}$$

If $\mathcal{H}(x) = \mathcal{H}(u(x))$, i.e. the right hand side depends only on u, we get

$$D_j\mathcal{H}D_ju = \frac{\partial \mathcal{H}}{\partial u}|Du|^2 \geq 0$$

since $\mathcal{H}(x,u)$ was assumed to be monotonically increasing in u. In total we get

$$D_i(\underbrace{a^{ik}(Du)}_{=:\widetilde{a}^{ik}(x)} D_k(|Du|^2)) \geq 0 \ .$$

By the weak maximum principle for weak solutions we get

$$\max_{\bar{\Omega}} |Du|^2 \leq \max_{\partial\Omega} |Du|^2.$$

We will now present a more geometric way of obtaining a maximum principle for the gradient, by computing om $M^n = \text{graph}(u) \subset \mathbb{R}^{n+1}$. First some facts from differential geometry: Let $\text{vec}(M)$ be the space of vectorfields on M, then the covariant derivative ∇ on M is a map

$$\nabla : \text{vec}(M) \times \text{vec}(M) \to \text{vec}(M),$$

with the following properties: Let $X, Y, Z \in \text{vec}(M)$ and $f,\ g \in C^\infty(M)$, then

$$\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z \ .$$

Additionally it should hold that

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y = f\nabla_X Y + df(X)Y.$$

On $\mathbb{R}^{n+1}$ we define

$$\big(\bar{\nabla}_X Y\big)_i = X(Y_i) \quad \text{i.e.} \quad (\bar{\nabla}_X Y) = \big(X(Y_1), X(Y_2), \ldots, X(Y_{n+1})\big).$$

One can easily check that the above structure conditions are fulfilled. Now let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface. One obtains the induced connection on M^n as follows: Let $X, Y \in \text{vec}(M)$, i.e.

$$X(p),\ Y(p) \in T_pM \quad \forall\, p \in M \text{ and define}$$

$$(\nabla_X Y)_p := \Pi_{T_pM}(\bar\nabla_X Y) = \bar\nabla_X Y - \left\langle \bar\nabla_X Y\,,\,\nu\right\rangle \nu,$$

with the outward unit normal ν, and one gets that

$$X\big(g(Y,Z)\big) = g(\bar\nabla X, Y) + g(X, \bar\nabla Y),$$

where g is the induced metric on M, i.e. $g(p)$ is the restriction of the Euclidean scalar product $\langle\cdot,\cdot\rangle$ to the tangent space T_pM for all $p \in M$.

As defined above, let ∇ be the covariant derivative on M corresponding to the induced metric. Let $e_1, \dots, e_n, \nu$ a local frame on $U \subset M^n = \operatorname{graph} u$. Then the second fundamental form $A = (h_{ij})$ is given by

$$h_{ij} = \left\langle \bar\nabla_{e_i}\nu\,,\,e_j\right\rangle = -\left\langle \nu\,,\,\bar\nabla_{e_i}e_j\right\rangle,$$

the mean curvature $H = tr_g(h_{ij}) = g^{ij}h_{ij}$, where (g_{ij}) is the metric on M^n, and (g^{ij}) is the inverse of the matrix (g_{ij}).

The upward pointing unit normal on $\operatorname{graph} u$ is given by

$$\nu = \frac{1}{\sqrt{1+|Du|^2}}(-Du, 1).$$

Let $v := \sqrt{1+|Du|^2} = \langle \nu\,,\,\tau_{n+1}\rangle^{-1}$, where τ_{n+1} is the unit vector in $(n+1)$-direction in $\mathbb{R}^{n+1}$. We have already defined the operator $\operatorname{div} : \operatorname{vec}(\mathbb{R}^{n+1})\big|_M \to \mathbb{R}$. Let $u \in C^\infty(M)$, then define

$$^M\Delta(u) := \operatorname{div}_M(\nabla^M u) \qquad \text{(Laplace-Beltrami)}$$

where $\nabla^M u = \operatorname{grad}^M(u) = g^{ij}\frac{\partial u}{\partial e_i}e_j$. In the case that u is defined in a neigbourhood in $\mathbb{R}^{n+1}$ one can check that

$$\begin{aligned}\nabla^M u &= \Pi_{T_pM}(Du) \qquad \text{(orthonormal projection to the tangent space)}\\ &= Du - \langle Du\,,\,\nu\rangle\,\nu.\end{aligned}$$

In Gaussian normal coordinates around p, that is at p we have $g(e_i, e_j) = g_{ij} = \delta_{ij}$ and $\nabla_{e_i}e_j = 0$, we can compute

$$\begin{aligned}
{}^M\Delta v &= \nabla_{e_i}(\nabla_{e_i} v)\\
&= \nabla_{e_i}\nabla_{e_i}\langle \nu\,,\,\tau_{n+1}\rangle^{-1}\\
&= \nabla_{e_i}(-v^2\left\langle \bar\nabla_{e_j}\nu\,,\,\tau_{n+1}\right\rangle) = \nabla_{e_i}(-v^2 h_{ij}\langle e_i, \tau_{n+1}\rangle)\\
&= -2v\nabla_{e_i}v\,\underbrace{h_{ij}\langle e_i\,,\,\tau_{n+1}\rangle}_{=-v^{-2}\nabla_{e_i}v} - v^2 \underbrace{\nabla_{e_i}h_{ij}}_{=\nabla_{e_j}h_{ii}=\nabla_j H}\langle e_j\,,\,\tau_{n+1}\rangle - v^2 h_{ij}\left\langle \underbrace{\bar\nabla_{e_i}e_j}_{-h_{ij}\nu}\,,\,\tau_{n+1}\right\rangle\\
&= 2v^{-1}\left|\nabla^M v\right|^2 - v^2\left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle + v^2\underbrace{h_{ij}h_{ji}}_{|A|^2}\underbrace{\langle \nu\,,\,\tau_{n+1}\rangle}_{v^{-1}}\\
&= 2v^{-1}\left|\nabla^M v\right|^2 + v|A|^2 - v^2\left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle.
\end{aligned}$$

So we obtain:

$$ {}^{M}\Delta v = 2v^{-1}\left|\nabla^M v\right|^2 + v|A|^2 - v^2\left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle \tag{6.5} $$

We now compute $\langle \nabla H\,,\,\tau_{n+1}\rangle$. We extend H constantly in τ_{n+1}-direction, i.e. we write

$$ H(p) = H((x,z)) = \tilde{H}(x). $$

For $g : \mathbb{R}^{n+1} \to \mathbb{R}$ we have

$$ \nabla^M g = \bar{\nabla} g - \left\langle \bar{\nabla} g\,,\,\nu\right\rangle \nu = Dg - \langle Dg\,,\,\nu\rangle\,\nu = \Pi_{T_pM}(Dg) $$

Thus we have

$$ \left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle = \underbrace{\left\langle D\tilde{H}\,,\,\tau_{n+1}\right\rangle}_{=0} - \left\langle D\tilde{H}\,,\,\nu\right\rangle \langle \nu\,,\,\tau_{n+1}\rangle = \sum_{i=1}^{n} D_i\tilde{H} D_i u v^{-2}\ . $$

Note that by our choice of normal vector and our sign convention we have that $\mathcal{A}u = \mathcal{H}(x,u)$ implies that $\tilde{H}(x) = -\mathcal{H}(x,u(x))$. So we get

$$ -v^2\left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle = -\sum_{i=1}^{n} D_i\tilde{H} D_i u = \sum_{i=1}^{n} \frac{\partial \mathcal{H}}{\partial x_i} D_i u + \frac{\partial \mathcal{H}}{\partial u}|Du|^2\,. $$

In the case that $\mathcal{H}(x,u) = \mathcal{H}(u)$ with $\frac{\partial \mathscr{H}}{\partial u} \geq 0$ we thus have

$$ {}^{M}\Delta v = 2v^{-1}|\nabla v|^2 + v|A|^2 + \frac{\partial \mathscr{H}}{\partial u}|Du|^2 \geq 0 $$

which implies by the maximum principle on M^n that

$$ \sup_M v \leq \sup_{\partial M} v \quad \Longleftrightarrow \quad \sup_{\Omega} \sqrt{1+|Du|^2} \leq \sup_{\partial\Omega} \sqrt{1+|Du|^2}. $$

We will now consider the case

$$ \frac{\partial \mathcal{H}}{\partial x_i} \neq 0\,. $$

To do this we multiply (6.5) with $e^{\lambda u}$:

$$ e^{\lambda u}\ {}^{M}\Delta v = 2v^{-1}e^{\lambda u}|\nabla v|^2 + e^{\lambda u}|A|^2 v - v^2\left\langle \nabla^M H\,,\,e_{n+1}\right\rangle e^{\lambda u}, $$

and compute with $g := e^{\lambda u} v$

$$ \begin{aligned} {}^{M}\Delta g &= \nabla_{e_i}\nabla_{e_i}(e^{\lambda u} v) = \nabla_{e_i}(\lambda e^{\lambda u} v \nabla_{e_i} u + e^{\lambda u}\nabla_{e_i} v) \\ &= \lambda\langle \nabla g\,,\,\nabla u\rangle + \lambda g \Delta u + \underbrace{\lambda e^{\lambda u}\nabla_{e_i} u \nabla_{e_i} v}_{\lambda\langle \nabla g\,,\,\nabla u\rangle - \lambda^2 v e^{\lambda u}|\nabla u|^2} + e^{\lambda u}\ {}^{M}\Delta v \\ &= 2\lambda\langle \nabla g\,,\,\nabla u\rangle + \lambda e^{\lambda u} v\ {}^{M}\Delta u - \lambda^2 v e^{\lambda u}|\nabla u|^2 + e^{\lambda u}\ {}^{M}\Delta v \\ &= 2\lambda\langle \nabla g\,,\,\nabla u\rangle + \lambda e^{\lambda u} v\ {}^{M}\Delta u - \lambda^2 v e^{\lambda u}|\nabla u|^2 + 2v^{-1}e^{\lambda u}|\nabla v|^2 + e^{\lambda u}|A|^2 v \\ &\quad - v^2 e^{\lambda u}\left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle \end{aligned} $$

Note that

$$\begin{aligned} 2v^{-1}e^{\lambda u}|\nabla v|^2 &= 2v^{-1}\langle \nabla g\,,\,\nabla v\rangle - 2\lambda e^{\lambda u}\langle \nabla u\,,\,\nabla v\rangle \\ &= 2v^{-1}\langle \nabla g\,,\,\nabla v\rangle - 2\lambda\langle \nabla u\,,\,\nabla g\rangle + 2\lambda^2 e^{\lambda u} v|\nabla u|^2\,, \end{aligned}$$

so we get

(6.6)

$$^M\Delta g = 2v^{-1}\langle \nabla g\,,\,\nabla v\rangle + \lambda^2 e^{\lambda u} v|\nabla u|^2 + \lambda e^{\lambda u} v\; ^M\Delta u + e^{\lambda u}|A|^2 v - v^2 e^{\lambda u}\left\langle \nabla^M H\,,\,\tau_{n+1}\right\rangle\,.$$

To compute derivatives of u we proceed as follows

- We can write $u = \langle \vec{x}\,,\,\tau_{n+1}\rangle = x_{n+1}$, where $\vec{x}$ is the position vector. This gives

$$\nabla_{e_i} u = \langle e_i\,,\,\tau_{n+1}\rangle$$

  and

$$\begin{aligned} |\nabla^M u|^2 &= \sum_{i=1}^n \langle e_i\,,\,\tau_{n+1}\rangle^2 \\ &= \left|\Pi_{T_pM}(\tau_{n+1})\right|^2. \\ &= 1 - \langle \nu\,,\,\tau_{n+1}\rangle^2 = 1 - v^{-2}. \end{aligned}$$

- For $^M\Delta u$ we can compute

$$\begin{aligned} ^M\Delta u &= \nabla_{e_i}(\nabla_{e_i} u) = \nabla_{e_i}(\langle e_i\,,\,\tau_{n+1}\rangle) = \left\langle \bar{\nabla}_{e_i} e_i\,,\,\tau_{n+1}\right\rangle \\ &= -\underbrace{h_{ii}}_{=H}\langle \nu\,,\,\tau_{n+1}\rangle = -Hv^{-1}. \end{aligned}$$

  Inserting this in (6.6) yields

$$^M\Delta g = 2v^{-1}\langle \nabla g\,,\,\nabla v\rangle + \lambda^2 g\left(1 - \frac{1}{v^2}\right) - \lambda e^{\lambda u} H + e^{\lambda u}|A|^2 v - v^2 e^{\lambda u}\left\langle \nabla^M H\,,\,e_{n+1}\right\rangle.$$

We now assume that $\frac{\partial H}{\partial z} \geq 0$, but $\frac{\partial H}{\partial x^i} \neq 0$. We assume further that $\sup_\Omega |u| \leq M$. We had already seen before that we can estimate, with $C_1 = \sup_{\Omega\times[-M,M]} |D_x\mathcal{H}|$, that

$$-v^2\left\langle \nabla^M H\,,\,e_{n+1}\right\rangle = \sum_{i=1}^n \frac{\partial \mathcal{H}}{\partial x_i} D_i u + \frac{\partial \mathcal{H}}{\partial z}|Du|^2 \geq -C_1|Du| \geq -C_1 v\,.$$

This implies

$$\Delta g \geq 2v^{-1}\langle \nabla g, \nabla v\rangle + \lambda^2 g\left(1 - \frac{1}{v^2}\right) - \lambda H\frac{g}{v} + |A|^2 g - C_1 g$$

By Cauchy-Schwartz we can estimate

$$|H| = |\langle(\lambda_1, \ldots, \lambda_n), (1, \ldots, 1)\rangle| \leq (\lambda_1^2 + \ldots + \lambda_n^2)^{1/2} n^{1/2}$$

which is equivalent to $|A|^2 \geq \frac{1}{n}H^2$. With the estimate

$$g\frac{\lambda}{v}H \leq g\left(\frac{n\lambda^2}{4v^2} + \frac{1}{n}H^2\right)$$

we get at a point where $v^2 \geq n+4$, that

$$\begin{aligned}\Delta g &\geq 2v^{-1}\langle\nabla g, \nabla v\rangle + \frac{\lambda^2}{2}g - \lambda^2 g\left(\frac{n+4}{4v^2}\right) + g|A|^2 - g\frac{1}{n}H^2 - C_1 g\\ &\geq 2v^{-1}\langle\nabla g, \nabla v\rangle + \frac{\lambda^2}{4}g - C_1 g + g\underbrace{(|A|^2 - \frac{1}{n}H^2)}_{\geq 0}\\ &\geq 2v^{-1}\langle\nabla g, \nabla v\rangle + g\end{aligned}$$

if we choose $\lambda = \sqrt{4C_1+4}$. So putting this together, this implies that g cannot attain an interior maximum, if

$$v \geq \sqrt{n+4}\ \text{, i.e.}\quad g \geq \sqrt{n+4}\, e^{(4C_1+4)u}$$

This implies

$$\sup_\Omega g \leq \max\left\{\sup_{\partial\Omega} g, \sqrt{n+4}\, e^{(4C_1+4)M}\right\}$$

Note that

$$\sup_{\partial\Omega} g = \sup_{\partial\Omega}\sqrt{1+|Du|^2}e^{(4C_1+4)u} \leq e^{(4C_1+4)M}\sup_{\partial\Omega}\sqrt{1+|Du|^2},$$

and thus

$$\begin{aligned}|Du| \leq \sqrt{1+|Du|^2} &\leq e^{-(4C_1+4)u}g \leq e^{(4C_1+4)M}\sup_\Omega g\\ &\leq e^{(8C_1+8)M}\max\left\{\sup_{\partial\Omega}\sqrt{1+|Du|^2}, \sqrt{n+4}\right\},\end{aligned} \tag{6.7}$$

where $M = \sup_\Omega |u|$ and $C_1 = \sup_{\Omega\times[-M,M]}|D_x\mathcal{H}|$.

6.3 Construction of barriers for surfaces of prescribed mean curvature

In this section let $\Omega \subset \mathbb{R}^n$ be bounded, with $\partial\Omega$ of class C^2. We assume that the boundary values φ on $\partial\Omega$ are given by $\varphi \in C^2(\bar{\Omega})$. If the boundary values are only given as $\varphi \in C^2(\partial\Omega)$ we extend them to the interior. We fix throughout a point $x_0 \in \partial\Omega$.

Let $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\Gamma_\mu = \{x \in \overline{\Omega}, d(x) < \mu\}$ for some $\mu > 0$. We aim to construct barriers $\delta^\pm : \bar{\Gamma}_\mu \to \mathbb{R}$ which fulfill

$$\mathcal{A}\delta^- \geq \mathcal{H}(x, \delta^-)\,, \qquad \mathcal{A}\delta^+ \leq \mathcal{H}(x, \delta^+)$$

and $\delta^-(x_0) = \varphi(x_0) = \delta^+(x_0)$ as well as

$$\begin{aligned} \delta^-(y) = \varphi(y) = \delta^+(y) &\quad \forall\, y \in \partial\Omega \\ \delta^+(y) \geq u(y) \geq \delta^-(y) &\quad \forall\, y \in \partial\Gamma_\mu \cap \Omega \end{aligned}$$

Assuming that we have constructed these barriers $\delta^\pm$ satisfying the above conditions, with bounded gradient in x_0, then we argue as follows. Given a solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of $\mathcal{A}u = \mathcal{H}(x, u)$ with boundary values φ we write

$$Qu = \mathcal{A}u - \mathcal{H}(x, u) = 0$$

and

$$Q\delta^+ = \mathcal{A}\delta^+ - \mathcal{H}(x, \delta^+) \leq 0,$$

so $Q\delta^+ \leq Qu$ with $\delta^+ \geq u$ on $\partial\Gamma_\mu$. We can check that the necessary conditions to apply Theorem 4.2.4 are fulfilled and we obtain that $u \leq \delta^+$ on Γ_μ. Similarly we see that $u \geq \delta^-$ on Γ_μ. Furhtermore, $\delta^-(x_0) = \varphi(x_0) = \delta^+(x_0)$. This implies

$$\frac{\delta^-(x) - \delta^-(x_0)}{|x - x_0|} \leq \frac{u(x) - u(x_0)}{|x - x_0|} \leq \frac{\delta^+(x) - \delta^+(x_0)}{|x - x_0|}$$

and with $x \to x_0$ that

$$|Du(x_0)| \leq \max(|D\delta^+|(x_0), |D\delta^-|(x_0))\,. \tag{6.8}$$

To construct such bariers we will try the following strategy: We set $\delta^\pm = \varphi \pm \psi(d)$, where d is the distance function to the boundary $\partial\Omega$ and ψ is chosen suitably with $\psi(0) = 0$.

To show the existence of such barriers we need a condition between the mean curvature of the boundary, $H(\partial\Omega)$, and the mean curvature of the graph of u, $H = -\mathcal{H}(x, u(x))$. To motivate this, think of graph $\delta^\pm$ beeing 'squeezed' in the cylinder $\partial\Omega \times \mathbb{R}$.

Let us assume that there is no gradient bound, i.e. there is a solution u of $\mathcal{A}u = \mathcal{H}(x, u)$ with $u = \varphi$ on $\partial\Omega$ and

$$D_\nu u(x_0) = +\infty\ .$$

We furthermore assume that graph u is still C^2 written as a graph over $(\partial\Omega \times \mathbb{R}) \cap B_R^{n+1}(x_0)$. We then have

$$H(\partial\Omega \times \mathbb{R})\big|_{(x_0, \varphi(x_0))} = H(\partial\Omega)\big|_{x_0}(x_0) \leq H(\operatorname{graph} u)\big|_{(x_0, \varphi(x_0))} = -\mathcal{H}(x_0, \varphi(x_0))$$

Assuming that $D_\nu u(x_0) = -\infty$ we get

$$H(\partial\Omega \times \mathbb{R})\big|_{(x_0,\varphi(x_0))} = H(\partial\Omega)\big|_{x_0}(x_0) \leq -H(\operatorname{graph} u)\big|_{(x_0,\varphi(x_0))} = \mathcal{H}(x_0, \varphi(x_0))$$

So it turns out that a necessary and sufficient condition to prevent this and to get the existence of barriers is to assume that

$$H_{\partial\Omega}\big|_y > |\mathcal{H}(y, \varphi(y))| \quad \forall\, y \in \partial\Omega\ . \tag{6.9}$$

To compute $A\delta^+ = A\varphi + A\psi(d)$ we have to understand the distance function to the boundary $d(\cdot)$.

Let $\Omega \subset \mathbb{R}^n$ be bounded. Then as we have seen in the exercises $d(x) = \operatorname{dist}(x, \partial\Omega)$ is Lipschitz continuous, with Lipschitz constant one.

To continue further we additionally assume that ν is the inner unit normal to $\partial\Omega$. After a rotation of the coordinate system we can identify $\nu(x_0)$ with e_n.

Then $\partial\Omega$ can be written in a neighbourhood U of x_0 as the graph of a function $\gamma \in C^2(\{x_n = (x_0)_n\} \cap U)$, that is

$$\partial\Omega \cap U = \operatorname{graph}\gamma(x')\,,$$

where $x' = (x_1, \ldots, x_{n-1})$ and $D\gamma(x_0') = 0$. Recall that

$$\nu = (-D\gamma, 1)\frac{1}{\sqrt{1 + |D\gamma|^2}}$$

and with the tangent vectors $\hat{e}_i = \frac{\partial F}{\partial x_i'}$, where F is the embedding $(x_1', \ldots, x_{n-1}') \mapsto (x_1', \ldots, x_{n-1}', \gamma(x_1', \ldots, x_{n-1}'))$ we have

$$h_{ij} = \langle \bar\nabla_{\hat{e}^i}\hat{e}_j, \nu\rangle = -\langle \hat{e}_j, \bar\nabla_{\hat{e}_i}\nu\rangle = -\langle \hat{e}_j, \frac{\partial(\nu\circ\gamma)}{\partial x_i'}\rangle\ .$$

Thus in this coordinate system we have at x_0 that

$$\nabla_{\hat{e}^i}\hat{e}_j = \Big(0, \ldots, \frac{\partial^2\varphi}{\partial x_i'\partial x_j'}\Big)$$

which implies that

$$h_{ij}(x_0) = D_i D_j \gamma(x_0')\ ,$$

and the eigenenvalues of $D^2\gamma$ in y_0' are just the principal curvatures $(\lambda_1, \ldots, \lambda_n)$. So we can rotate the coordinate system around e_n such that $D^2\gamma$ is diagonal with the principal curvatures as entries, that is

$$D_{ij}\gamma(x_0') = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1})\,.$$

Note that then also in this coordinate system, at this point

$$\frac{\partial}{\partial x'_j}\big(\nu_i \circ \gamma\big)\Big|_{x'=x'_0} = -\sum_i \lambda_i \delta_{ij}$$

for $i, j = 1, \ldots, n-1$. We call such a coordinate system a principal curvature system at x_0.

Theorem 6.3.1. *Let Ω be bounded and $\partial\Omega$ of class C^k for $k \geqq 2$. Then there exists $\mu > 0$, depending on Ω, such that $d \in C^k(\Gamma_\mu)$ where $\Gamma_\mu = \{x \in \overline{\Omega}, d(x) < \mu\}$.*

Proof. Since $\partial\Omega$ is of class C^2 and Ω bounded, $\partial\Omega$ satisfies a uniform interior sphere condition. Let us assume that the radius of theses spheres is bounded below by $\mu > 0$ beschränkt. (Clearly $\lambda_i \leq \frac{1}{\mu}$, $i = 1, \ldots, n-1$). Then for every point $x \in \Gamma_\mu$, there exists a unique point $y(x) \in \partial\Omega$, such that $|x - y| = d(x)$. Then we have

$$x = y(x) + \nu(y(x))d(x) = y + \nu(y)d(x)\,.$$

We now fix $x_0 \in \Gamma_\mu$, $y_0 = y(x_0)$ and choose a principal coordinate system in y_0 for $\partial\Omega$, where again $\partial\Omega$ is given locally in a neighbourhood U of y_0 as the graph of γ. We define $g : (\{x_n = (y_0)_n\} \cap U) \times \mathbb{R} \to \mathbb{R}^n$ by

$$g(y', d) = y + \nu(y)d, \qquad \text{where } y = (y', \gamma(y'))\,. \tag{6.10}$$

Then $g \in C^{k-1}$ and in (y'_0, d) it holds:

$$Dg = \operatorname{diag}(1 - \lambda_1 d, \ldots, 1 - \lambda_{n-1}d, 1)$$
$$\det(Dg)_{(y'_0, d(x_0))} = (1 - \lambda_1 d)\ldots(1 - \lambda_{n-1}d) > 0$$

since $d(x_0) < \mu$ and $\lambda_i \leq \frac{1}{\mu}$. By the inverse mapping theorem, in a neighbourhood M of x_0, we have that g is a local diffeomorphism and the mapping $x \mapsto y'$ is in C^{k-1}. From (6.10) we have that

$$Dd(x) = \nu(y(x)) = \nu(\varphi(y'(x))) \in C^{k-1}(M) \forall x \in M\,.$$

This gives $d \in C^k(M)$ which implies $d \in C^k(\Gamma_\mu)$. □

Lemma 6.3.2. *Let Ω and μ satisfy the conditions as above and let $x_0 \in \Gamma_\mu$ and $y_0 \in \partial\Omega$ as before, i.e, $d(x_0) = |x_0 - y_0|$. Then in terms of a principal coordinate system at y_0 we have*

$$(D^2 d(x_0)) = \operatorname{diag}\left[\frac{-\lambda_1}{1 - \lambda_1 d}, \ldots, \frac{-\lambda_{n-1}}{1 - \lambda_{n-1}d}, 0\right]$$

Proof. Since

$$Dd(x_0) = \nu(y_0) = (0, \ldots, 0, 1)$$

and

$$Dd(x) = \nu(y_0) \quad \forall x = y_0 + \nu(y_0)d$$

we have

$$\Rightarrow D_i D_n d(x_0) = D_n D_i d(x_0) = 0, i = 1, \ldots, n$$

To obtain the other derivatives we compute for $i, j = 1, \ldots, n-1$:

$$\begin{aligned} D_i D_j d(x)_{|x=x_0} = D_j(\nu_i(y(x)))_{|x=x_0} &= D_k \nu_i(y_0) D_j y_k(x_0) \\ &= \begin{cases} \frac{-\lambda_i}{1-\lambda_i d} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \end{aligned}$$

since

$$(Dg) = \operatorname{diag}(1 - \lambda_1 d, \ldots, 1 - \lambda_{n-1} d, 1)$$

and

$$D_j \nu_i(\varphi(y')) = -\sum_i \lambda_i \delta_{ij} \,.$$

□

Remark 6.3.3: *(i) Up to topological restrictions we can choose* μ *as:*

$$\mu = \frac{1}{\max_{i=1,\ldots,n-1, x\in\partial\Omega}\{\lambda_i(x)\}}$$

(ii) This statement is equivalent to the observation that the circles of principal curvatures of $\partial\Omega$ *at* y_0 *and the ones of the equidistant surface to* $\partial\Omega$ *through* x_0 *are concentric.*

We now return to the construction of suitable barriers. Recall that we made the Ansatz $\delta^{\pm} = \varphi \pm \psi(d) : \Gamma_\mu \to \mathbb{R}$, where d is the distance function to the boundary $\partial\Omega$ and $\psi : \mathbb{R} \to \mathbb{R}$ is chosen suitably with $\psi(0) = 0$. We can furthermore assume that the boundary values φ are such that $\varphi(y) = \varphi(y_0)$ for all $y \in \Gamma_{\mu/2}$ and $y_0 \in \partial\Omega$ such that $d(y) = |y - y_0|$. We will start with the computation for δ^+. Recall that we can write $\mathcal{A}\delta^+ = a^{ij}(D\delta^+)D_{ij}\delta^+$ with

$$a^{ij}(D\delta^+) = \frac{1}{\sqrt{1 + |D\delta^+|^2}}\Big(\delta^{ij} - \frac{D^i\delta^+ D^j\delta^+}{1 + |D\delta^+|^2}\Big) \,.$$

We have

$$D_j\delta^+ = D_j\varphi + \psi' D_j d \quad \text{and} \quad D_{ij}\delta^+ = D_{ij}\varphi + \psi'' D_i d\, D_j d + \psi' D_{ij} d \,.$$

Note that $|Dd|^2 = 1$ implies that $D_{ij}d\, D^i d = 0$ and we can compute

$$\begin{aligned} a^{ij} D_{ij} d &= \frac{1}{\sqrt{1 + |D\delta^+|^2}}\Delta d - \frac{(D^i\varphi + \psi' D^i d)(D^j\varphi + \psi' D^j d)}{(1 + |D\delta^+|^2)^{3/2}} D_{ij} d \\ &= \frac{1}{\sqrt{1 + |D\delta^+|^2}}\Delta d - \frac{D^i\varphi D^j\varphi}{(1 + |D\delta^+|^2)^{3/2}} D_{ij} d \,. \end{aligned}$$

Note that since $|Dd| = 1$ on Γ_μ we have by Lemma 6.3.2 for $y \in \Gamma_\mu$ that

$$\Delta d(y) = \sum_{i=1}^{n-1} \frac{-\lambda_i}{1 - \lambda_i d} \leq -\sum_{i=1}^{n-1} \lambda_i = -H(y_0),$$

where $y_0 \in \partial\Omega$ is such that $d(y) = |y_0 - y|$, the λ_i are the principal curvatures of $\partial\Omega$ at y_0 and $H(y_0)$ is the mean curvature of $\partial\Omega$ at y_0. We can then estimate at y

$$\begin{aligned}\mathcal{A}\delta^+ \leq & \frac{1}{\sqrt{1+|D\delta^+|^2}}|D^2\varphi| + \psi'' a^{ij} D_i d\, D_j d - \frac{\psi'}{\sqrt{1+|D\delta^+|^2}} H(y_0) \\ & - \frac{\psi' D^i\varphi D^j\varphi}{(1+|D\delta^+|^2)^{3/2}} D_{ij} d\end{aligned}$$

We make that Ansatz $\psi(d) = \frac{1}{\nu}\log(1+kd)$ where k, ν are positive constants to be chosen later. We then have

$$\psi' = \frac{k}{\nu(1+kd)} \quad \text{and} \quad \psi'' = -\frac{k^2}{\nu}\frac{1}{(1+kd)^2}$$

Note that $\psi'' < 0$. So assume $d(y) < \mu/2$ and we can estimate

$$\begin{aligned}\mathcal{A}\delta^+ \leq & \frac{1}{\sqrt{1+|D\delta^+|^2}}|D^2\varphi| - \frac{k^2}{\nu}\frac{1}{(1+kd)^2}\frac{1}{(1+|D\delta^+|^2)^{3/2}} \\ & - \frac{k}{\nu(1+kd)}\frac{1}{\sqrt{1+|D\delta^+|^2}} H(y_0) + C_1 \frac{\|\varphi\|^2_{C^1}}{(1+|D\delta^+|^2)^{3/2}}\frac{k}{\nu(1+kd)}\end{aligned} \tag{6.11}$$

where $\max_{i=1,\ldots n-1} \frac{\lambda_i}{1-\lambda_i d} \leq 2/\mu =: C_1$. To compensate the last term with half of the second term on the right hand side we need

$$\frac{k}{(1+kd)} \geq 2C_1\|\varphi\|^2_{C^1} \ .$$

Note that we have for $d \leq \eta_1$

$$\frac{k}{(1+kd)} = \frac{1}{(1/k+d)} \geq \frac{1}{(1/k+\eta_1)}.$$

So if we assume that

$$k \geq 2C_1\|\varphi\|^2_{C^1} \quad \text{and} \quad \eta_1 \leq \max\{\mu/2, (4C_1\|\varphi\|^2_{C^1})^{-1}\} \tag{6.12}$$

we can achive this. To continue we estimate

$$1+|D\delta^+|^2 \leq 1 + 2|D\varphi|^2 + 2(\psi')^2 \leq 1 + 2\|\varphi\|^2_{C^1} + 2\frac{k^2}{\nu^2(1+kd)^2} \ .$$

We now want to use what is left of the second term to compensate the first term on the right hand side of (6.11). To do this it thus suffices to have

$$\frac{1}{2}\frac{k^2}{\nu(1+kd)^2} \geq \|\varphi\|_{C^2}\Big(1+2\|\varphi\|^2_{C^1}+2\frac{k^2}{\nu^2(1+kd)^2}\Big)\,. \tag{6.13}$$

We first choose

$$\nu = 8\|\varphi\|^2_{C^2}\ , \tag{6.14}$$

this implies that it suffices to show (6.13), to ask for

$$\frac{k^2}{(1+kd)^2} \geq 4\nu\|\varphi\|_{C^2}(1+2\|\varphi\|^2_{C^1}) = 32\|\varphi\|^2_{C^2}(1+2\|\varphi\|^2_{C^1}) := \gamma\,.$$

Again we have for $d \leq \eta_2$ that

$$\frac{k^2}{(1+kd)^2} = \frac{1}{(\frac{1}{k}+d)^2} \geq \frac{1}{(\frac{1}{k}+\eta_2)^2}$$

So if we assume that

$$k \geq 2\gamma^{1/2} \quad \text{and} \quad \eta_2 \leq \max\{\mu/2, \gamma^{-1/2}/2\} \tag{6.15}$$

this is as well fulfilled. Now by our main assumption (6.9) there exists an $\varepsilon_0 > 0$ such that

$$H_{\partial\Omega}(z) \geq -\mathcal{H}(z,\varphi(z)) + \varepsilon_0 \quad \forall\, z \in \partial\Omega\,.$$

Now let $\eta = \min\{\eta_1, \eta_2\}$. We have shown that on Γ_η it holds

$$\mathcal{A}\delta^+ \leq -\frac{k}{\nu(1+kd)}\frac{1}{\sqrt{1+|D\delta^+|^2}}H(y_0)\ . \tag{6.16}$$

We want to show now that the factor in front of $H(y_0)$ gets arbitrarily close to one, provided η is small enough and k big. Recall that we have

$$\begin{aligned} 1+|D\delta^+|^2 &= 1+|D\varphi|^2+(\psi')^2+2\psi'\langle D\varphi, Dd\rangle \\ &= 1+|D\varphi|^2+\Big(\frac{k}{\nu(1+kd)}\Big)^2+2\frac{k}{\nu(1+kd)}\langle D\varphi, Dd\rangle \end{aligned}$$

This implies that

$$\Big(\frac{\nu(1+kd)}{k}\Big)^2(1+|D\delta^+|^2) = 1+2\Big(\frac{\nu(1+kd)}{k}\Big)\langle D\varphi, Dd\rangle + \Big(\frac{\nu(1+kd)}{k}\Big)^2(1+|D\varphi|^2)\,,$$

and note that

$$0 \leq \frac{\nu(1+kd)}{k} \leq \nu\Big(\frac{1}{k}+\eta\Big) \to 0$$

for $k \to \infty$ and $\eta \to 0$. This implies that for

$$\eta \le \eta_3 \quad \text{and} \quad k > k_0 \tag{6.17}$$

where η_3 and k_0 only depend on $\|\varphi\|_{C^1}$ and $\sup_{\partial\Omega} |H_{\partial\Omega}|$ we have that

$$\mathcal{A}\delta^+(y) \le -H(y_0) + \frac{\varepsilon_0}{4} \le \mathcal{H}(y_0, \varphi(y_0)) - \frac{3}{4}\varepsilon_0 \le \mathcal{H}(y_0, \delta^+(y)) - \frac{3}{4}\varepsilon_0\,, \tag{6.18}$$

since $\frac{\partial \mathcal{H}}{\partial z} \ge 0$ and $\delta^+(y) \ge \varphi(y_0)$. Since $|D_x\mathcal{H}(x,z)| \le C_2$ for $x \in \Omega$ and $z \in [-\sup_\Omega |u|, \sup_\Omega |u|]$ we have that for $\eta \le \eta_4$ that

$$\mathcal{H}(y_0, \delta^+(y)) - \frac{3}{4}\varepsilon_0 \le \mathcal{H}(y, \delta^+(y))$$

for all $y \in \Gamma_\mu$. So we now fix

$$\eta = \min\{\eta_1, \eta_2, \eta_3, \eta_4\},$$

and choose $k \ge \max\{2C_1\|\varphi\|^2_{C^1}, 2\gamma^{1/2}, k_0\}$ such that for every $y \in \partial\Gamma_\eta \cap \Omega$

$$\delta^+(y) = \varphi(y_0) + \frac{1}{\nu}\log(1 + k\eta) \ge \sup_\Omega |u|$$

we have that

$$\mathcal{A}\delta^+(y) \le \mathcal{H}(y, \delta^+) \tag{6.19}$$

on Γ_η, together with $\delta^+ = \varphi$ on $\partial\Omega$ and $\delta+ \ge \sup_\Omega |u|$ on $\partial\Gamma_\eta \cap \Omega$. By replacing u by $-u$ and $\mathcal{H}(x,z)$ by $-\mathcal{H}(x,-z)$ we can similarly construct a lower barrier δ^-. We can collect this in the following proposition.

Proposition 6.3.4. *Let Ω be bounded with $\partial\Omega$ of class C^2, $\varphi \in C^2(\partial\Omega)$, and $\mathcal{H}$ in $C^1(\bar\Omega \times \mathbb{R})$ with $\frac{\partial}{\partial z}\mathcal{H} \ge 0$. Assume that*

$$H_{\partial\Omega}(y) > |\mathcal{H}(y, \varphi(y))| \quad \forall\, y \in \partial\Omega\,.$$

Then for given $M > 0$ there exist $\nu, k, \eta > 0$, depending only on Ω, $\|\varphi\|_{C^2(\partial\Omega)}$, $\|D_x\mathcal{H}\|_{L^\infty(\Omega\times[-M,M])}$ and M such that

$$\delta^\pm := \tilde\varphi + \frac{1}{\nu}\log(1 + kd)\,,$$

where $\tilde\varphi \in C^2(\bar\Omega)$ is a suitable extension of φ, satisfy

$$\mathcal{A}\delta^+ \le \mathcal{H}(\cdot, \delta^+) \quad \textit{and} \quad \mathcal{A}\delta^- \ge \mathcal{H}(\cdot, \delta^-)$$

on Γ_η with

$$\delta^+ \ge M \quad \textit{and} \quad \delta^- \le -M$$

on $\partial\Gamma_\eta \cap \Omega$.

6.4 Hölder estimates for the gradient

We have so far proven a-priori estimates for solutions of $\mathcal{A}u = \mathcal{H}(\cdot, u)$ in $C^1(\bar\Omega)$. The missing estimate to apply Theorem 4.2.1 is a Hölder estimate for Du. To do this we aim to apply the DeGiorgi-Nash-Moser estimates to Du. So let us assume that $\partial\Omega$ is of class $C^{2,\alpha}$ and $u \in C^{2,\alpha}(\bar\Omega)$ is a solution of

$$\mathcal{A}u = D_i A^i(Du) = \mathcal{H}(\cdot, u)\ . \tag{6.20}$$

Differentiating this in direction e_k, we can argue completely analogously as in Lemma 4.1.4 that $v := D_k u \in C^{1,\alpha}(\bar\Omega)$ is a weak solution of

$$D_i(a^{ij}(Du)D_j v) = D_k(\mathcal{H}(\cdot, u(\cdot)) = D_i f^i$$

in Ω where $a^{ij}(Du)$ is given as in Lemma 4.1.4 and we set $f^i := 0$ if $i \neq k$ and $f^i = \mathcal{H}(\cdot, u(\cdot))$ for $i = k$. We have

$$\sup_i \|f^i\|_{\infty;\Omega} \leq \sup_{\Omega\times[-M,M]} \mathcal{H}\,,$$

where $M = \sup_\Omega |u|$. Note that the coefficients a^{ij} are bounded and uniformly elliptic whith constants $\lambda, \Lambda > 0$, dependig only on $\sup_\Omega |Du|$. This enables us to apply Theorem 5.3.6 to get:

Lemma 6.4.1. *Let Ω be a bounded domain, with $\partial\Omega$ of class $C^{2,\alpha}$ and $u \in C^{2,\alpha}(\Omega)$ be a solution of*

$$\mathcal{A}u = \mathcal{H}(\cdot, u)$$

where $\mathcal{H} : \bar\Omega \times \mathbb{R}$ is continuous. Given $\Omega' \Subset \Omega$, there are constants $C = C(n, \Omega', \Omega, \|u\|_{C^1(\Omega)}, \sup_\Omega |\mathcal{H}(\cdot, u)|)$ and $\alpha' = \alpha'(n, \Omega', \Omega, \|u\|_{C^1(\Omega)}) > 0$ such that

$$\|u\|_{C^{1,\alpha'}(\Omega')} \leq C\ .$$

Unfortunately we cannot apply Theorem 5.4.4 directly to obtain a $C^{1,\alpha'}$-estimate for u on the whole set Ω. This is due to the fact that from the boundary values φ we only get oscillation bounds for derivatives parallel to the boundary.

To compensate for this fact, we first locally straighten out the boundary: Since Ω is precompact and $\partial\Omega$ is of class $C^{2,\alpha}$ there is $R > 0$ and $C_1 > 0$ such that for every $x_0 \in \partial\Omega$ there is a diffeomorphism $\psi : \Omega \cap B_R(x_0) \to B_R^+(0)$ with the properties that $\partial\Omega \cap B_R(x_0)$ is mapped to $\{x_n = 0\} \cap B_R(0)$ and $\|\psi, \psi^{-1}\|_{C^{2,\alpha}} \leq C_1$. Define $\tilde u : B_R^+(0) \to \mathbb{R}$ by $\tilde u(\psi(x)) = u(x)$. From equation (6.20) we obtain (after a small calculation) that $\tilde u$ satisfies on $B_R^+(0)$:

$$D_i\Big(\beta^{-1}\tilde a^{ij}\frac{D_j\tilde u}{\sqrt{1+\tilde a^{rs}D_r\tilde u D_s\tilde u}}\Big) = \beta^{-1}\mathcal{H}(\psi^{-1}(\cdot), \tilde u)\,, \tag{6.21}$$

where $\tilde{a}^{ij} = D_k\psi^i(\psi^{-1}(\cdot))D^k\psi^j(\psi^{-1}(\cdot))$, which is a poisitve definite matrix and $\beta = \det(D\psi)(\psi^{-1}(\cdot))$. We fix $k \in \{1, \ldots, n-1\}$ and differentiate in direction e_k to obtain that $v = D_k\tilde{u}$ satisfies an equation of the form:

$$D_i\Big(\bar{\alpha}^{ij}D_jv\Big) = D_if^i_k \ , \tag{6.22}$$

where

$$\bar{\alpha}^{ij} = \frac{\beta^{-1}}{\sqrt{1+\tilde{a}^{rs}D_r\tilde{u}D_s\tilde{u}}}\Big(\tilde{\alpha}^{ij} - \frac{\tilde{\alpha}^{is}D_s\tilde{u}\,\tilde{\alpha}^{jr}D_r\tilde{u}}{1+\tilde{a}^{rs}D_r\tilde{u}D_s\tilde{u}}\Big)$$

and

$$f^i_k = f^i_k(x, \tilde{u}, D\tilde{u}, \mathcal{H}, \psi) \ ,$$

which is bounded, provided $|u|$ and $|Du|$ are bounded. It is easily checked that $\bar{\alpha}^{ij}$ is uniformly elliptic, provided $|D\tilde{u}|$ is bounded. We furthermore have that

$$v = D_k\tilde{u} = D_k\tilde{\varphi}$$

on $\{x_n = 0\} \cap B_R(0)$, where $\tilde{\varphi} = \varphi \circ \psi^{-1}$. Since we have assumed that $\varphi \in C^{2,\alpha}(\partial\Omega)$ we obtain a $C^{0,\alpha}$-bound for v on this boundary portion. Using Theorem 5.4.3 and arguing now as in Theorem 5.4.4 we see that there exists an $\alpha' > 0$ such that

$$[D_k\tilde{u}]_{\alpha';B^+_{3R/4}(0)} \leq C \tag{6.23}$$

where $C = C(n, \sup_\Omega |u|, \sup_\Omega |Du|, \mathcal{H}, \psi)$ and $k = 1, \ldots, n-1$.

It remains to prove the estimate for $k = n$. We first replace $\tilde{u}$ by $\bar{u} = \tilde{u} - \tilde{\varphi}$. Then (6.21) implies that $\bar{v} = D_k\bar{u}$ satisfies

$$D_i\big(\bar{\alpha}^{ij}D_j\bar{v}\big) = D_if^i_k - D_i\big(\bar{\alpha}^{ij}D_j(D_k\tilde{\varphi})\big) \ , \tag{6.24}$$

Since we assumed that $\varphi \in C^2(\bar{\Omega})$ this implies that

$$D_i\big(\bar{\alpha}^{ij}D_j\bar{v}\big) = D_i\bar{f}^i_k \, , \tag{6.25}$$

with $\bar{f}^i_k \in L^\infty(B^+_R(0))$ and $\bar{v} = D_k\bar{u} = 0$ on $\{x_n = 0\} \cap B_R(0)$. To get the estimate we want to apply the following lemma.

Lemma 6.4.2 (Morrey)**.** *Let $u \in W^{1,1}(\Omega)$ and assume that there exist $K > 0$, $0 < \alpha \leq 1$ such that*

$$\int_{B_R} |Du|\,dx \leq KR^{n-1+\alpha} \quad \forall B_R \subset \Omega \, . \tag{6.26}$$

Then $u \in C^{0,\alpha}(\Omega)$, and for any ball $B_R \subset \Omega$

$$\operatorname{osc}_{B_R(x)} u \leq CR^\alpha \ , \tag{6.27}$$

where $C = C(n, \alpha)$. If $\Omega = \tilde{\Omega} \cap R^n_+$ for some domain $\tilde{\Omega} \subset \mathbb{R}^n$ and (6.26) holds for all balls $B_R \subset \tilde{\Omega}$, then $u \in C^{0,\alpha}(\bar{\Omega} \cap \tilde{\Omega})$ and (6.27) holds for all $B_R \subset \tilde{\Omega}$.

Proof. Combine Lemma 5.2.4 and Lemma 5.2.2 as in the proof of Corollary 5.2.3. □

Now for $y_0 \in B_{3R/4}(0)$ choose a cut-off function $\eta,\ 0 \le \eta \le 1$ such that

$$\eta(y) = \begin{cases} 1 & \text{if } y \in B_r(y_0) \\ 0 & \text{if } y \notin B_{2r}(y_0) \end{cases}$$

with $|D\eta| \le 2/r$. We further assume that $B_{2r}(y_0) \Subset B_{3R/4}(0)$. Since $\bar{v} = D_k\bar{u} = 0$ on $\{x_n = 0\} \cap B_R(0)$ we can extend v by zero to all of $B_R(0)$. Then let

$$\zeta = \eta^2(\bar{v} - c),$$

where we choose

$$c = \begin{cases} 0 & \text{if } B_{2r}(y_0) \cap B_R^-(0) \neq \emptyset \\ \bar{v}(y_0) & \text{if } B_{2r} \subset B_R^+(0)\ . \end{cases}$$

Then ζ is in $W_0^{1,2}(B_R^+(0))$, so we have by (6.25) that

$$\int_{B_R^+(0)} \bar{\alpha}^{ij} D_j\bar{v} D_i\zeta\, dy = \int_{B_R^+(0)} \bar{f}_k^i D_i\zeta\, dy\,,$$

which implies

$$\int_{B_R^+(0)} \eta^2 \bar{\alpha}^{ij} D_j\bar{v} D_i\bar{v}\, dy \le \int_{B_R^+(0)} 2\eta|(\bar{v} - \bar{v}(y_0))\bar{\alpha}^{ij} D_i\eta D_j\bar{v}| + \eta^2|\bar{f}_k^i D_i\bar{v}| + 2\eta|(\bar{v} - c)\bar{f}_k^i D_i\eta|\, dy\ .$$

By Young's inequality, together with the uniform ellipticity of $\bar{\alpha}^{ij}$ this implies

$$\int_{B_R^+(0)} \eta^2|D\bar{v}|^2\, dy \le C \int_{B_R^+(0)} \big(\eta^2 + |D\eta|^2(\bar{v} - c)^2\big)\, dy\,,$$

where $C = C(n, \Omega, \psi, \|u\|_{C^1}, \|\varphi\|_{C^2(\partial\Omega)}, \mathcal{H})$. By the choice of η this implies

$$\int_{B_r^+(y_0)} |D\bar{v}|^2\, dy \le Cr^{n-2}\big(r^2 + \sup_{B_{2r}(y_0)} (\bar{v} - c)^2\big) \le Cr^{n-2+2\alpha'}\,,$$

since by (6.23)

$$\sup_{B_{2r}(y_0)} (\bar{v} - c)^2 \le C' r^{2\alpha'}\ .$$

Since $\bar{u} = \tilde{u} - \tilde{\varphi}$, we obtain for $j \neq n$ that

$$\int_{B_r^+(y_0)} |D_{ij}\tilde{u}|^2 \, dy \leq C r^{n-2+2\alpha'} \,. \tag{6.28}$$

By computing the divergence in (6.21), as in (6.22), $\tilde{u}$ solves

$$\bar{\alpha}^{ij} D_{ij}\tilde{u} = \mathcal{H}(\cdot, \tilde{u}) + b(\cdot, D\tilde{u}),$$

where b depends on ψ and $D\tilde{u}$ and is bounded, provided $|D\tilde{u}|$ is bounded. Solving this equation for $D_{nn}\tilde{u}$ we get

$$D_{nn}\tilde{u} = \frac{1}{\bar{\alpha}^{nn}} \Big(\mathcal{H}(\cdot, \tilde{u}) + b(\cdot, D\tilde{u}) - \sum_{\substack{i,j \\ (i,j) \neq (n,n)}} \bar{\alpha}^{ij} D_{ij}\tilde{u} \Big).$$

Since $\bar{\alpha}^{ij}$ is uniformly elliptic, we have $\bar{\alpha}_{nn} \geq \bar{\lambda} > 0$ and we get from (6.29) that

$$\int_{B_r^+(y_0)} |D_{nn}\tilde{u}|^2 \, dy \leq C r^{n-2+2\alpha'} \,.$$

This implies that

$$\int_{B_r^+(y_0)} |D_i(D_n\tilde{u})|^2 \, dy \leq C r^{n-2+2\alpha'}$$

for $i = 1, \ldots n$ and all $B_{2r}(y_0) \subset B_{3R/4}(0)$. Theorem 6.4.2, together with Hölder's inequality, then gives that also

$$[D_n\tilde{u}]_{\alpha'; B^+_{3R/4}(0)} \leq C \,,$$

which gives together with (6.23), and going back to u via ψ, assuming w.l.o.g. that $\|\psi, \psi^{-1}\|_{C^{2,\alpha'}}$ is sufficently small, that

$$[Du]_{\alpha'; B_{R/2}(x_0) \cap \Omega} \leq C \,.$$

Covering $\partial\Omega$ with such balls B_R and using Lemma 6.4.1 we arrive at:

Proposition 6.4.3. *Let Ω be a bounded domain, with $\partial\Omega$ of class $C^{2,\alpha}$ and $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution of*

$$\mathcal{A}u = \mathcal{H}(\cdot, u)$$

where $\mathcal{H} : \bar{\Omega} \times \mathbb{R}$ is continuous and $u = \varphi$ on $\partial\Omega$ with $\varphi \in C^2(\partial\Omega)$. Then there are constants $C = C\big(n, \Omega, \partial\Omega, \|u\|_{C^1(\Omega)}, \sup_\Omega |\mathcal{H}(\cdot, u)|), \|\varphi\|_{C^2(\partial\Omega)}\big)$ and $\alpha' = \alpha'\big(n, \Omega, \partial\Omega, \|u\|_{C^1(\Omega)}, \|\varphi\|_{C^2(\partial\Omega)}\big) > 0$ such that

$$\|u\|_{C^{1,\alpha'}(\Omega)} \leq C \,.$$

6.5 Existence of surfaces of prescribed mean curvature

We can now put everything together to obtain the following existence and uniqueness result.

Theorem 6.5.1 (Giusti, Serrin). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domian, with $\partial\Omega$ of class $C^{2,\alpha}$ and let $\mathcal{H} : \bar{\Omega}\times\mathbb{R}$ be of class C^1 with $\frac{\partial}{\partial z}\mathcal{H} \geq 0$, $\frac{\partial}{\partial z}\mathcal{H}$ of class $C^{0,\alpha}$ and $\varphi \in C^{2,\alpha}(\partial\Omega)$. Suppose that there is an $\varepsilon_0 > 0$ such that*

$$(i) \qquad \Big|\int_\Omega \mathcal{H}(x,0)\eta(x)\,dx\Big| \leq (1-\varepsilon_0)\int_\Omega |D\eta|\,dx \qquad \forall\,\eta \in C^1_c(\Omega)\,,$$

$$(ii) \qquad H_{\partial\Omega}(x) > |\mathcal{H}(x,\varphi(x))| \qquad \forall\,x \in \partial\Omega\,,$$

then the Dirichlet problem $\mathcal{A}u = \mathcal{H}(\cdot,u)$, $u|_{\partial\Omega} = \varphi$ has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$.

Proof. We aim to apply the method of continuity, Theorem 4.2.1. To do this we have to show that for any solution $v \in C^{2,\alpha}(\bar{\Omega})$ of

$$\mathcal{A}v = \mathcal{H}(\cdot,v), \qquad v|_{\partial\Omega} = \varphi$$

there are positive constants $C = C(\Omega,\mathcal{H},\|\varphi\|_{C^{2,\alpha}(\bar{\Omega})})$, $\alpha' = \alpha'(\Omega,\mathcal{H},\|\varphi\|_{C^{2,\alpha}(\bar{\Omega})})$ such that

$$(6.29) \qquad \|v\|_{C^{1,\alpha'}(\bar{\Omega})} \leq C\;.$$

The sup-estimate: By condition (i) we can apply Proposition 6.1.2 to see that we get an estimate

$$(6.30) \qquad \sup_\Omega |v| \leq C_1\,,$$

where $C_1 = C_1(n,\varepsilon_0^{-1},|\Omega|,\sup_{\partial\Omega}|\varphi|)$.

The gradient esimate: By Proposition 6.3.4 and condition (ii) we obtain the existence of upper and lower barriers, where we choose M to be the bound given in (6.30). As explained in the introduction of section 6.3 this implies that

$$(6.31) \qquad \sup_{\partial\Omega} |Dv| \leq C_2\,,$$

where C_2 depends only on $n, C_1, \Omega, \|\varphi\|_{C^2(\partial\Omega)}, \|D_x\mathcal{H}\|_{L^\infty(\Omega\times[-C_1,C_1])}$ and condition (ii). Then the maximum principle for the gradient, (6.7), implies that

$$(6.32) \qquad \sup_\Omega |Dv| \leq C_3\,,$$

where C_3 depends only on the same quantities as C_2.

The Hölder estimate for the gradient: By Proposition 6.4.3 together with (6.30) and (6.32) there exists positive constants C_4 and α' such that

$$\|v\|_{C^{1,\alpha'}(\Omega)} \leq C_4 \,,$$

where C_4, α' depend only on the same quantities as C_1 and C_3. Especially they only depend on $\Omega, \mathcal{H}$ and $\|\varphi\|_{C^2(\Omega)}$.

This establishes the existence of a solution $u \in C^{2,\alpha}(\bar{\Omega})$. Uniqueness follows from Lemma 4.2.2 or alternatively from Theorem 4.2.4. $\square$

Bibliography

[1] J. Dieudonné, *Grundzüge der modernen Analysis, Band 1*, third ed., Logik und Grundlagen der Mathematik [Logic and Foundations of Mathematics], vol. 8, Friedr. Vieweg & Sohn, Braunschweig, 1985, Translated from the English by Ludwig Boll and Klaus Matthes.

[2] Lawrence C. Evans, *Partial differential equations*, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[3] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[4] Leon Simon, *Schauder estimates by scaling*, Calc. Var. Partial Differential Equations **5** (1997), no. 5, 391–407.