Uniqueness of asymptotically conical tangent flows

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Overview

(1) Huisken's monotonicity formula and tangent flows

(2) The problem of unique limits

(3) Uniqueness of tangent flows in MCF, statement of main results

(4) Outline of proof idea

Monotonicity formula and tangent flows

Consider $(M_t^n)_{-T\leq t<0}$ a smooth mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$, i.e.

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} = -H\nu$$

for a smooth family $F(\cdot, t)$ of parametrisations. We assume that the flow forms a singularity at $(0,0)$.

Huisken's monotonicity formula

Consider the backwards heat kernel

$$\rho(x,t) = \frac{1}{(2\pi(-t))^{n/2}} e^{-\frac{|x|^2}{4(-t)}}$$

then

$$\frac{d}{dt}\int_{M_t} \rho \, d\mu = -\int_{M_t} \left|\vec{H} + \frac{x^{\perp}}{2(-t)}\right|^2 \rho \, d\mu \qquad (\star)$$

Monotonicity formula and tangent flows

Parabolic rescalings: For $\lambda > 0$ let

$$M_t^\lambda := \lambda M_{\lambda^{-2}t} \qquad t \in [-\lambda^2 T \leq t < 0)\,,$$

which is again a mean curvature flow.

Tangent flows: Consider $\lambda_i \to +\infty$, then subsequentially

$$(M_t^{\lambda_i})_{-\lambda_i^2 T \leq t < 0} \rightharpoonup (M_t')_{t<0}\,.$$

If the singularity is of type I (i.e. $|A| \leq C/(-t)^{1/2}$), then the convergence is smooth, otherwise as Brakke flows.

Huisken/Ilmanen/White: $(\star)$ implies that $(M_t')_{t<0}$ is self-similarly shrinking, i.e.

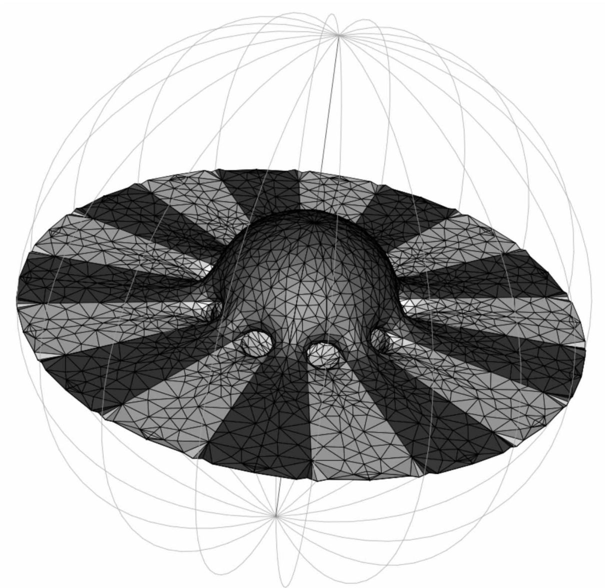
$$M_t' = \sqrt{-t} \cdot \Sigma$$

and Σ satisfies $\vec{H} = -\frac{x^\perp}{2}$. We call such a Σ a *self-shrinker*.

Monotonicity formula and tangent flows

Examples:

- Plane: $\mathbb{R}^n \subset \mathbb{R}^{n+1}$
- Sphere: $\mathbb{S}^n_{\sqrt{2n}} \subset \mathbb{R}^{n+1}$
- Cylinders: $\mathbb{S}^{n-k}_{\sqrt{2(n-k)}} \times \mathbb{R}^k \subset \mathbb{R}^{n+1}$ for $k = 1, \ldots, n-1$
- Huisken ('90): If $H \geq 0$ (which is preserved under the evolution), then these are the only possibilities.
- Angenent ('89): torus of revolution
- Ketover ('16): doublings of Platonic solids via min-max methods, discovered numerically by Chopp ('94)
- Møller ('14): desingularisation of 'Angenent torus' $\cup\, \mathbb{S}^2_2$
- Kapouleas-Kleene-Møller ('15): desingularisation of $\mathbb{R}^2 \cup \mathbb{S}^2_2$



Tom Ilmanen's conjectural shrinker of genus 8 with 9 Scherk handles

(picture used with his permission)

Monotonicity formula and tangent flows

Structure of self-shrinkers:

- $\lim_{\lambda \searrow 0} \lambda \cdot \Sigma = C_\infty$ asymptotic cone (in Hausdorff distance)
- We call Σ *asymptotically conical* if C_∞ and convergence smooth
- L. Wang:
  - C_∞ determines Σ uniquely (in the asymptotically conical case) ('14)
  - $\Sigma^2 \subset \mathbb{R}^3$ with finite genus $\Rightarrow \Sigma^2$ has only cylindrical or smoothly conical ends ('16)
- T. Ilmanen ('03): cylinder uniqueness conjecture

The problem of unique limits

Question: When is it true that the blow-up limit does not depend on the rescaling sequence?

Consider the rescaled flow

$$\widetilde{M}_\tau := \frac{1}{\sqrt{-t}} M_t \qquad \tau = -\log(-t), \quad \tau \in [-\log(T), \infty) \ .$$

Then

$$\Big(\frac{\partial \widetilde{F}}{\partial \tau}\Big)^\perp = \vec{H} + \frac{x^\perp}{2}$$

Monotonicity formula:

$$\frac{d}{d\tau} \int_{\widetilde{M}_\tau} \tilde{\rho}\, d\mu = -\int_{\widetilde{M}_\tau} \Big| \vec{H} + \frac{x^\perp}{2} \Big|^2 \tilde{\rho}\, d\mu \,,$$

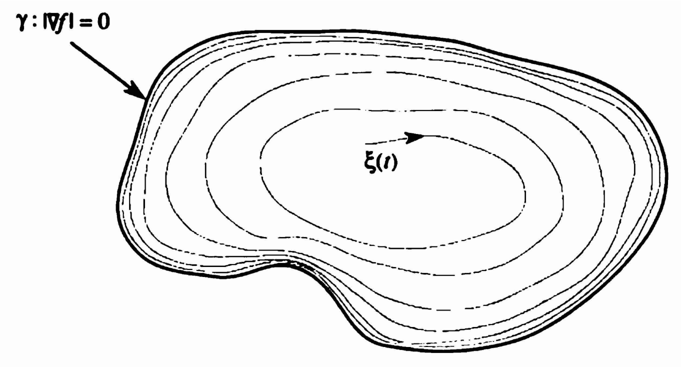
where $\tilde{\rho}(x) = (2\pi)^{-n/2} e^{-|x|^2/4}$.

So the above question is equivalent to asking if $\widetilde{M}_\tau$ converges to a unique limit shrinker when $\tau \to \infty$.

The problem of unique limits

Examples of non-uniqueness:

- L. Simon: there are examples of C^∞ functions f, with f and ∇f vanishing on a smooth Jordan curve γ, and such that there are solutions of $\xi' = -\nabla f(\xi)$, which 'spiral out towards γ' ("goat tracks down the hillside") as $t \to \infty$, so that the set of limit points is all of γ.



picture from L. Simon's ETH lecture notes

The problem of unique limits

- Consider the heat equation

$$u_t = u_{xx}$$

on $\mathbb{R}$. For a sequence of positive, strictly increasing radii $R_i \to \infty$ such that $R_i/R_{i+1} \to 0$ consider the initial data

$$u_0(x) = \sum_{i=1}^{\infty} (-1)^i \chi_{B_{R_{i+1}} \setminus B_{R_i}} \,.$$

We can write the solution as

$$u(x,t) = \int_{\mathbb{R}} u_0(y) \rho(x-y)\, dy \,.$$

One easily sees that although $|\nabla u| \leq C/\sqrt{t}$, there are subsequences (t_i) and (t_j) such that $u(\cdot, t_i) \to 1$ and $u(\cdot, t_j) \to -1$ as $t_i, t_j \to \infty$.

The problem of unique limits

Compare to uniqueness of tangent cones for minimal surfaces:

- Allard-Almgren ('81): unit multiplicity + isolated singularity + integrability of Jacobi fields of link
- White (83'): 2D + minimising
- Simon (83'): unit multiplicity + isolated singularity

Main tool Simon:

$$F(u) = \int_M E(x, u, \nabla u)$$

elliptic functional, analytic in $u, \nabla u$; (M, g) compact, u_0 critical point of F, ∇F the L^2-gradient.

Łojasiewicz–Simon inequality: $\exists\, \theta = \theta(M, u_0) \in (0, 1/2]$ such that

$$|F(u) - F(u_0)|^{1-\theta} \leq \|\nabla F(u)\|_{L^2(M)}$$

where $\|u - u_0\|_{C^{2,\alpha}(M)}$ sufficiently small.

Main tools: Finite dim. Łojasiewicz inequality + Liapunov-Schmidt reduction + Fredholm properties of the linearisation L of ∇F.

The problem of unique limits

Standard application LS-inequality:

$u(x,t)$ gradient flow of F, $u(\cdot,t_i) \to u_0$:

$$-\frac{d}{dt}\big(F(u(\cdot,t)) - F(u_0)\big)^{\theta} = \theta\big(F(u(\cdot,t)) - F(u_0)\big)^{\theta-1}\|\nabla F(u)\|^2_{L^2(M)}$$

$$\geq \theta\big(F(u(\cdot,t)) - F(u_0)\big)^{\theta-1}\|\nabla F(u)\|_{L^2(M)}\big\|\tfrac{\partial u}{\partial t}\big\|_{L^2(\Sigma)}$$

$$\geq \theta\big\|\tfrac{\partial u}{\partial t}\big\|_{L^2(\Sigma)}$$

Thus

$$\|u(t) - u(t_0)\|_{L^2(\Sigma)} \leq C\big(F(u(\cdot,t_0)) - F(u_0)\big)^{\theta}.$$

Uniqueness of tangent flows in MCF

We first note that Huisken's result that compact, convex hypersurfaces contract to round points implies the uniqueness of spherical tangent flows:

Theorem (Huisken '84): Let $\mathbb{S}^n_{\sqrt{2n}} \subset \mathbb{R}^{n+1}$ be the spherical self-shrinker. Let $\mathcal{M} = (M_t)_{-T \leq t < 0}$ be a MCF s.t. $\mathcal{M}_{\mathbb{S}^n_{\sqrt{2n}}} = (\sqrt{-t} \cdot \mathbb{S}^n_{\sqrt{2n}})_{t<0}$ arises as a unit multiplicity tangent flow to $\mathcal{M}$ at $(0,0)$. Then $\mathcal{M}_\Sigma$ is the unique tangent flow to $\mathcal{M}$ at $(0,0)$.

Theorem (S '14): Let $\Sigma^n \subset \mathbb{R}^{n+k}$ be a compact smooth self-shrinker. Let $\mathcal{M} = (M_t)_{-T \leq t < 0}$ be a MCF s.t. $\mathcal{M}_\Sigma = (\sqrt{-t} \cdot \Sigma)_{t<0}$ arises as a unit multiplicity tangent flow to $\mathcal{M}$ at $(0,0)$. Then $\mathcal{M}_\Sigma$ is the unique tangent flow to $\mathcal{M}$ at $(0,0)$.

Remark: This follows from a rather direct application of the LS-inequality to $F(M) = \int_M e^{-\frac{|x|^2}{4}}\, d\mu$ and an adaptation of the earlier described strategy.

Uniqueness of tangent flows in MCF

Theorem (Colding-Minicozzi '15): Let $\Sigma^n = \mathbb{S}^{n-l} \times \mathbb{R}^l \subset \mathbb{R}^{n+1}$ for some $l \in \{1, \ldots, n-1\}$. Let $\mathcal{M} = (M_t)_{-T \leq t < 0}$ be a MCF s.t. $\mathcal{M}_\Sigma = (\sqrt{-t} \cdot \Sigma)_{t<0}$ arises as a unit multiplicity tangent flow to $\mathcal{M}$ at $(0,0)$. Then $\mathcal{M}_\Sigma$ is the unique tangent flow to $\mathcal{M}$ at $(0,0)$.

Remark: The non-compactness of the shrinker $\Sigma^n = \mathbb{S}^{n-l} \times \mathbb{R}^l$ is one of the main stumbling blocks here: the rescaled flow $\widetilde{M}_\tau$ can never be written as an entire graph over Σ.

C-M directly prove a LS-inequality for $F(M) = \int_M e^{-\frac{|x|^2}{4}} \, d\mu$, using the special geometry of cylindrical shrinkers, together with a localisation and extension strategy.

Uniqueness of tangent flows in MCF

Theorem (Chodosh-S '19): Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be an asymptotically conical self-shrinker. Let $\mathcal{M} = (M_t)_{-T \leq t < 0}$ be a MCF s.t. $\mathcal{M}_\Sigma = (\sqrt{-t} \cdot \Sigma)_{t<0}$ arises as a unit multiplicity tangent flow to $\mathcal{M}$ at $(0,0)$. Then $\mathcal{M}_\Sigma$ is the unique tangent flow to $\mathcal{M}$ at $(0,0)$.

Uniqueness of tangent flows in MCF

Structure of singular set:

Theorem (C-M ('16/'18)): Let $\mathcal{M}$ be a (weak) mean curvature flow of hypersurfaces with only cylindrical tangent flows. Then the space-time singular set is contained in finitely many compact embedded $(n-1)$-dim Lipschitz-submanifolds + a $(n-2)$-dim set. (n=2: smooth for almost all times).

Theorem (Chodosh-S '19): Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be an asymptotically conical self-shrinker. Let $\mathcal{M} = (M_t)_{-T \leq t < 0}$ be a MCF s.t. $\mathcal{M}_\Sigma = (\sqrt{-t} \cdot \Sigma)_{t<0}$ arises as a unit multiplicity tangent flow to $\mathcal{M}$ at $(0,0)$. Then $\exists\, \varepsilon > 0$ s.t. $\forall t \in (-\varepsilon^2, 0)$, $M_t \cap B_\varepsilon(0)$ is diffeomorphic to Σ. As $t \nearrow 0$ $(M_t \cap (B_\varepsilon(0) \setminus \{0\}))$ converges in C^∞_{loc} to a smooth surface $M_0 \subset B_\varepsilon(0) \setminus \{0\}$ with a conical singularity, smoothly modelled on the asymptotic cone of Σ.

Remark: C-M ('12): planes, spheres and cylinders are the only entropy stable shrinkers $\rightsquigarrow$ possible way to construct a generic MCF.
Our result suggests that one can flow through such singularities instead of perturbing them away.

Proof idea

Setup: Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be an asymptotically conical self-shrinker. Let $\mathcal{M} = (M_t)_{-T \leq t < 0}$ be a MCF s.t. $\mathcal{M}_\Sigma = (\sqrt{-t} \cdot \Sigma)_{t<0}$ arises as a unit multiplicity tangent flow to $\mathcal{M}$ at $(0,0)$.

Main steps:

- Step I: LS-inequality for entire, asymptotically conical graphs over Σ
- Step II: localisation + extension: localised LS-inequality + error

Step I:

Let Γ be the link of C_∞. Consider Cone-Hölder spaces $\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)$ (compare K-K-M):

$u : \Sigma \to \mathbb{R}$ s.t. in coordinates $(r, \omega) \in (1, \infty) \times \Gamma$ along the end of Σ

$$u(r, \omega) = c(\omega) \cdot r + O(r^{-1})$$
$$\partial_r u(r, \omega) = c(\omega) + O(r^{-2})$$

Proof idea

Let

$$\phi := H - \langle x, \nu \rangle / 2$$

and consider its linearisation along Σ:

$$Lu = \Delta_\Sigma u - \frac{1}{2}(\vec{x} \cdot \nabla u - u) + |A_\Sigma|^2 u \ .$$

Note that L is self-adjoing w.r.t. the weight $\tilde{\rho} = (4\pi)^{-n/2} e^{-|x|^2/4}$. We show

- Schauder estimates for L in Cone-Hölder spaces
- regularity & existence (Fredholm) in L^2-based Sobolev spaces with weight $\tilde{\rho}$.

Outcome: L behaves as in the compact case considered by Simon ('83): Let $F(M) = \int_M \tilde{\rho} \, d\mu$ and $M = \mathrm{graph}_\Sigma(u)$. Then $\exists\, \theta \in (0, 1/2]$ s.t.

$$|F(M) - F(\Sigma)|^{1-\theta} \leq \left(\int_M |\phi|^2 \tilde{\rho} \, d\mu \right)^{\frac{1}{2}} \qquad (\star\star)$$

provided $\|u\|_{\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)}$ is sufficiently small.

Proof idea

Step II:

Consider again the rescaled flow

$$\widetilde{M}_\tau := \frac{1}{\sqrt{-t}} M_t \qquad \tau = -\log(-t), \quad \tau \in [-\log T, \infty) \ .$$

Can assume $\widetilde{M}_{\tau_i} \to \Sigma$ in C^∞_{loc} for some $\tau_i \to +\infty$.

Idea: Can assume $\widetilde{M}_{\tau_i} = \mathrm{graph}_\Sigma(u(\cdot, \tau_i))$ on $B_{\underline{r}}(0)$ for some $\underline{r}$ sufficiently large, with $\|u(\cdot, \tau_i)\|_{C^l(B_{\underline{r}}(0))}$ small. Call this the *core graphical hypothesis*.

Show that u extends to a function which is small in $\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)$. Apply $(\star\star)$ to $\mathrm{graph}_\Sigma(u)$ and control errors.

Proof idea

Note: the core graphical hypothesis is not sufficient to control the errors in the LS-inequality, since we must not destroy the good term

$$\int_{M_\tau} |\phi|^2 \rho \, d\mu \, . \qquad (\star\star\star)$$

Note

- cutting off $(\star\star\star)$ outside of $B_R(0)$ introduces errors of the order $o(1) \cdot e^{-\frac{R^2}{4}}$
- this motivates to define the *shrinker scale* $R(M_\tau)$ (compare C-M) via

$$e^{-\frac{R^2(M_\tau)}{4}} := \int_{M_\tau} |\phi|^2 \rho \, d\mu \, ,$$

thus any errors coming from cutting off at $R(\widetilde{M}_\tau)$ can still be compensated by $(\star\star\star)$.

Proof idea

- Assume we can make the LS-inequality work via cut-off and extension, such that the errors introduced can still be estimated by the RHS of $(\star\star)$.
- Assume further that the core graphical hypothesis is fulfilled for all $\tau \in [\tau_0, \bar{\tau})$.

Then we can estimate

$$-\frac{d}{dt}\big(F(\widetilde{M}_\tau) - F(\Sigma)\big)^{\theta} = \theta\big(F(\widetilde{M}_\tau) - F(\Sigma)\big)^{\theta-1} \int_{\widetilde{M}_\tau} |\phi|^2 \tilde{\rho}$$
$$\geq C\left(\int_{\widetilde{M}_\tau} |\phi|^2 \tilde{\rho}\right)^{1/2}$$
$$\geq C\left\|\frac{\partial u}{\partial \tau}\right\|_{L^2(\Sigma \cap B_{\underline{r}}(0))}.$$

Integrating this implies that the core graphical hypothesis is fulfilled beyond $\bar{\tau}$.

Proof idea

How to control the errors when cutting off the LS-inequality?

Goal: Show that $\widetilde{M}_\tau$ is graphical over Σ on $B_R(0)$ for $R \backsim R(\widetilde{M}_\tau)$. More precisely, show that $\exists\, u : \Sigma \to \mathbb{R}$, $\|u\|_{\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)}$ small s.t.

$$\widetilde{M}_\tau \cap B_R(0) \subset \mathrm{graph}_\Sigma(u)\,.$$

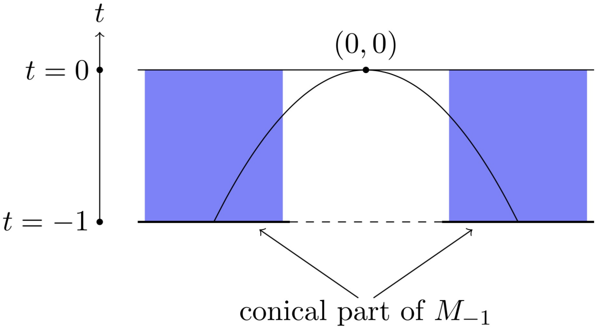
- We call the biggest such R the *conical scale* $r(\widetilde{M}_\tau)$.

As a first coarse scale, call

- the biggest R such that $\widetilde{M}_\tau$ is a graph over Σ and the curvature behaves like on a cone the *rough conical scale* $\tilde{r}(\widetilde{M}_\tau)$.

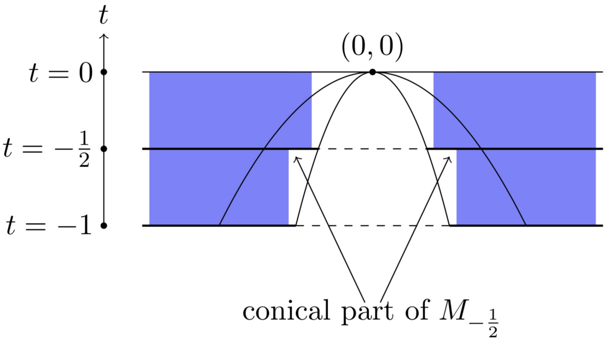
Proof idea

Claim: the rough conical scale improves rapidly, as long as the core graphical hypothesis is fulfilled.



The conical nature of the shrinker Σ (and thus the unrescaled flow at time $t = -1$) yields – via pseudolocality – curvature estimates in the region that is shaded blue. We can only expect the LS-inequality to give useful bounds *below* the parabola, since this is the set where ρ is uniformly bounded away from zero.

Proof idea



Assuming that we have control over M_t via the LS-inequality inside of the wide parabola (for $t \in [-1, \frac{1}{2})$), we can then use pseudolocality out of the conical region in $M_{-\frac{1}{2}}$ to gain curvature estimates on a *larger* region (still shaded blue).

Outcome: We can assume that $\tilde{r}(\widetilde{M}_\tau) \geq e^{\tau/2}\underline{r}/2$.

Proof idea

Final step: want to show that $R(\widetilde{M}_\tau) \backsim r(\widetilde{M}_\tau)$. Recall

$$\int_{\widetilde{M}_\tau} |\phi|^2 \rho \, d\mu = e^{-\frac{R^2(\widetilde{M}_\tau)}{4}} \ll 1\,.$$

By interpolation with the curvature estimates from the rough conical scale (compare C-M) we can assume that

$$\phi = H - \frac{\langle x, \nu \rangle}{2} \approx 0$$

on $B_{R(\widetilde{M}_\tau)}(0)$, which implies

$$\partial_r u \approx r^{-1} \langle x, \nu \rangle = O(r^{-2})\,.$$

- this yields asymptotic conicality
- show that smallness of u on the core can be extended outwards